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Random-Order Alternating Schwarz for Sparse Triangular Solves

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Sparse Triangular Systems

- **Occur for approximate incomplete factorization preconditioners**
  - Low solution accuracy required as **LU ≈ A** typically only a rough approximation.
  - Replace forward/backward substitutions with iterative method.
  - Better scalability of iterative methods.

- **Jacobi iteration**

$$x^{k+1} = D^{-1}\left(b - (A - D)x^k\right)$$
$$x^{k+1} = D^{-1}b + Mx^k$$

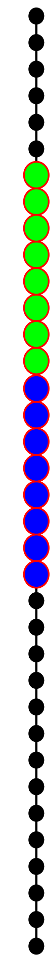
$$M_L = D_L^{-1}\left(D_L - L\right) = I - L$$
$$M_U = D_U^{-1}\left(D_U - U\right) = I - D_U^{-1}U$$

Sparse Triangular Systems

- **Block-Decomposition**
  - Typically, no information about the problem discretization.
  - Matrix partitioning, block sizes match hardware characteristics.
  - Over-decomposition for GPUs.

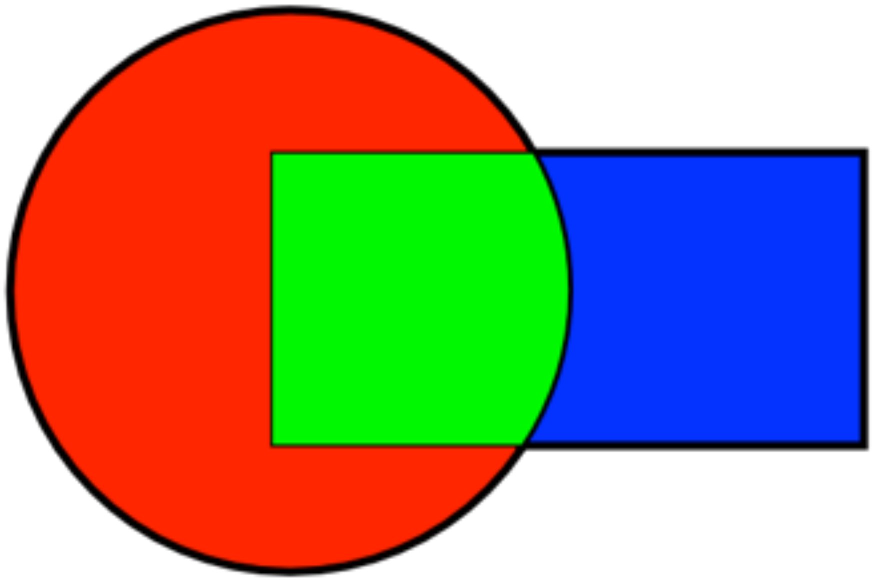
- **Clear information dependency**
  - Synchronous top-down subdomain scheduling results in (block-) substitution. For blocks containing one unknown, exact solve.
  - **Propagation** of **new information** in dependency direction is key.
  - Faster convergence expected for the scheduling:
    top-down in **Ly=b** and bottom-up in **Ux=y**.

  - GPUs do in general **not allow** to control the **scheduling**.



Domain Overlap Strategies

- **Global domain** is decomposed into **subdomains**.
- **A local problem** is solved for each subdomain.
- **Iterative process** generates the **global solution**.
- **Subdomains overlap** for faster **information propagation**.



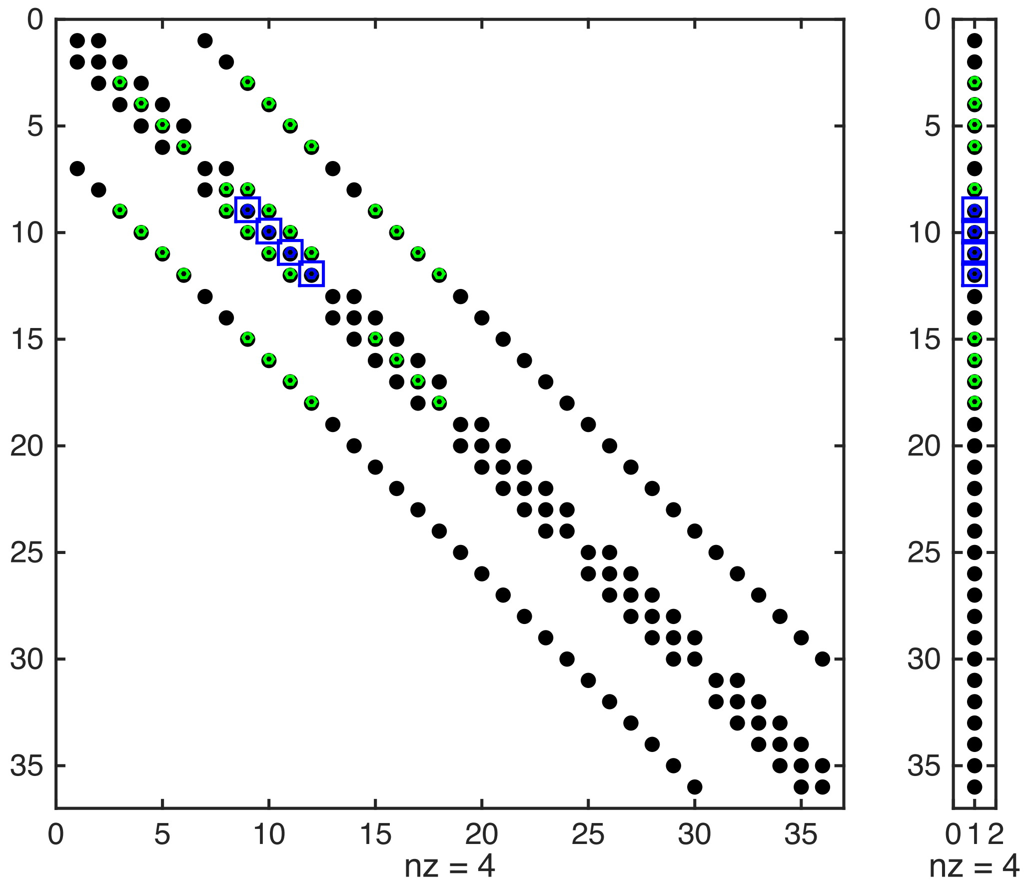
https://en.wikipedia.org/wiki/Domain_decomposition_methods

Domain Overlap Strategies

- **Alternating Schwarz**
  - Write back results for extended subdomain.
  - Sequential updates or multi-color ordering.
  - Fixed subdomain scheduling.
- **Restricted Additive Schwarz**
  - Write back results only for original subdomain.
  - Parallel update of all subdomains.
- **Random-Order Alternating Schwarz**
  - Write back results only for original subdomain.
  - Sequential subdomain updates.
  - Random subdomain scheduling.

Domain Overlap

- **Domain overlap based on matrix partitioning**
  - Blocks are extended by components adjacent in the matrix.



Local solver is applied to problem on extended subdomain.

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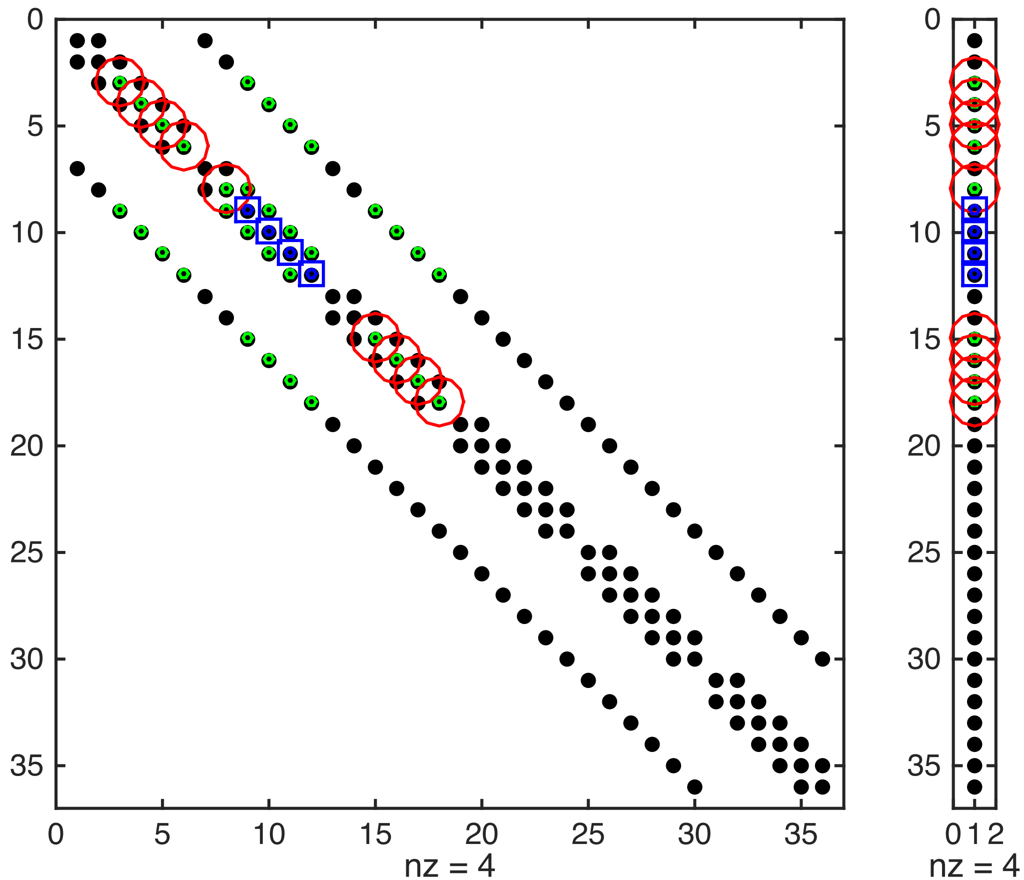

L-1 overlap

Local solver is applied to problem on extended subdomain.

Domain Overlap

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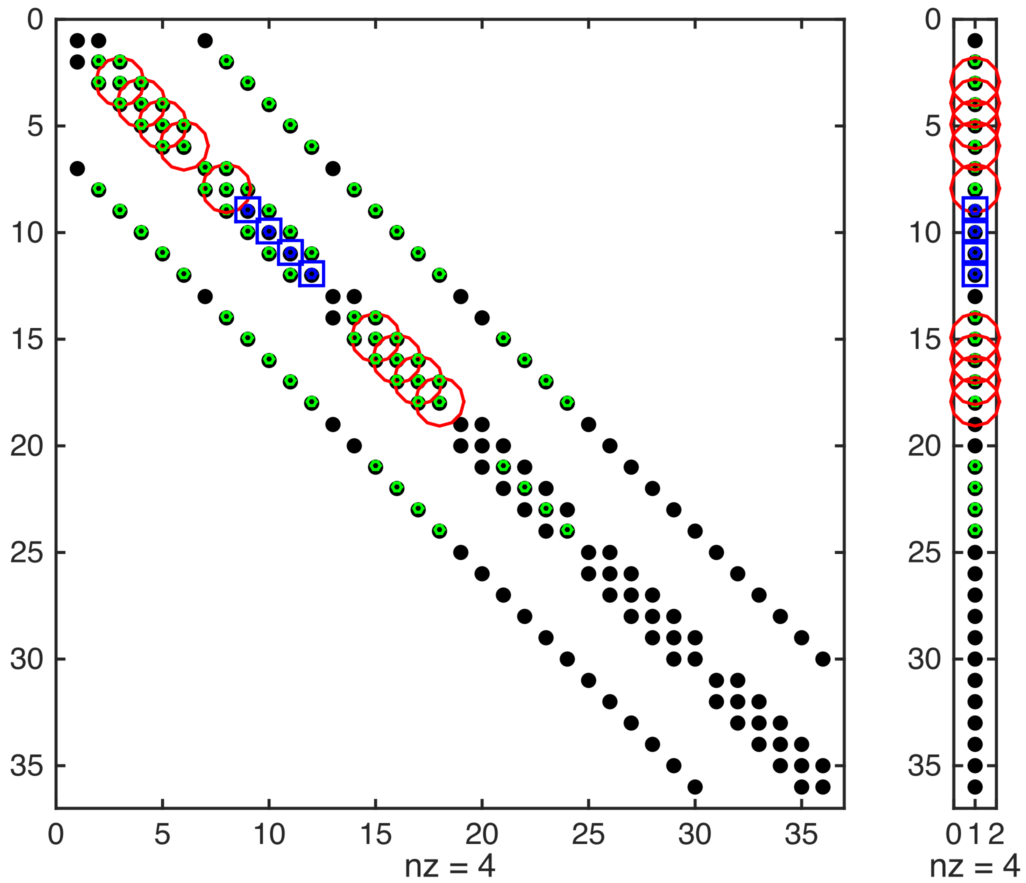

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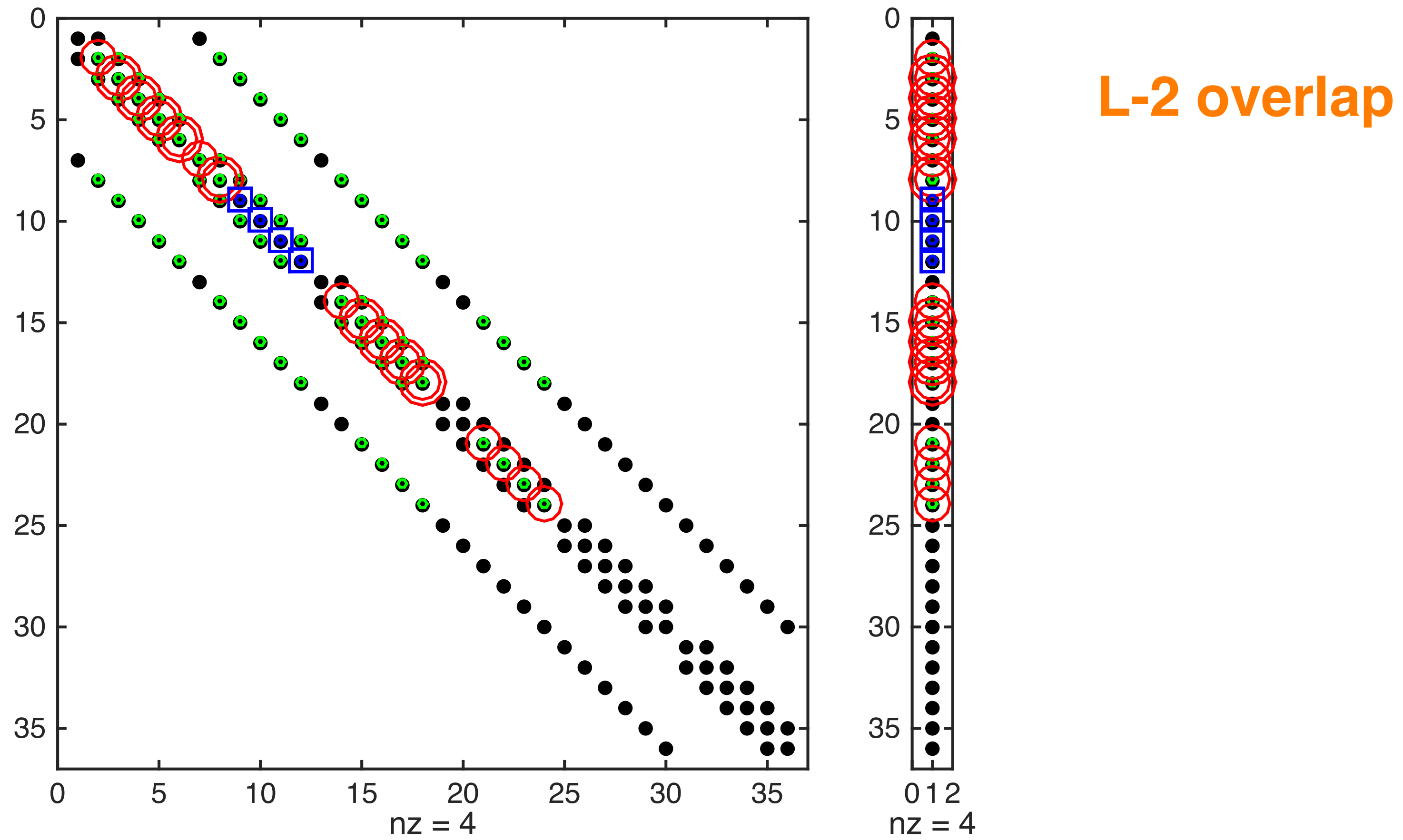


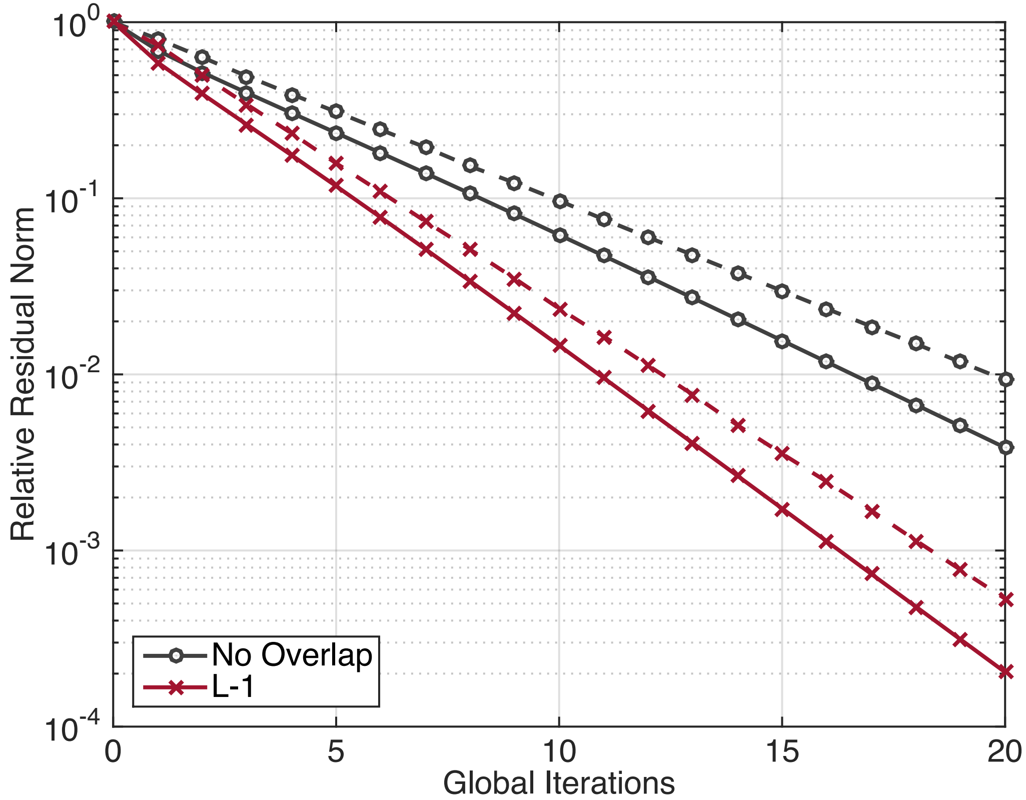
Local solver is applied to problem on extended subdomain.

Experiment Outline

- **Target Problems**
  - **Laplace Problem**, 3D, 27-point stencil, 8x8x8 grid, 47 blocks.
  - **Sparse triangular** systems from ILU(0) preconditioning.
- **Solver Setting**
  - 2 Jacobi sweeps as local solver on the blocks (subdomains).
  - Different update schemes in-between subdomains:
    - **Fixed Gauss-Seidel** top-down subdomain scheduling.
    - **Random** (subdomains are updated once per global iteration).
- **Overlap**
  - **Uniform** overlap derived from matrix partitioning.
  - **Non-uniform / directed** overlap derived from matrix characteristics.
- **Analyze convergence.**
  - **Normalized iterations** account for overhead of overlap.

Alternating Schwarz Convergence

Test case: L3D 27-pt stencil



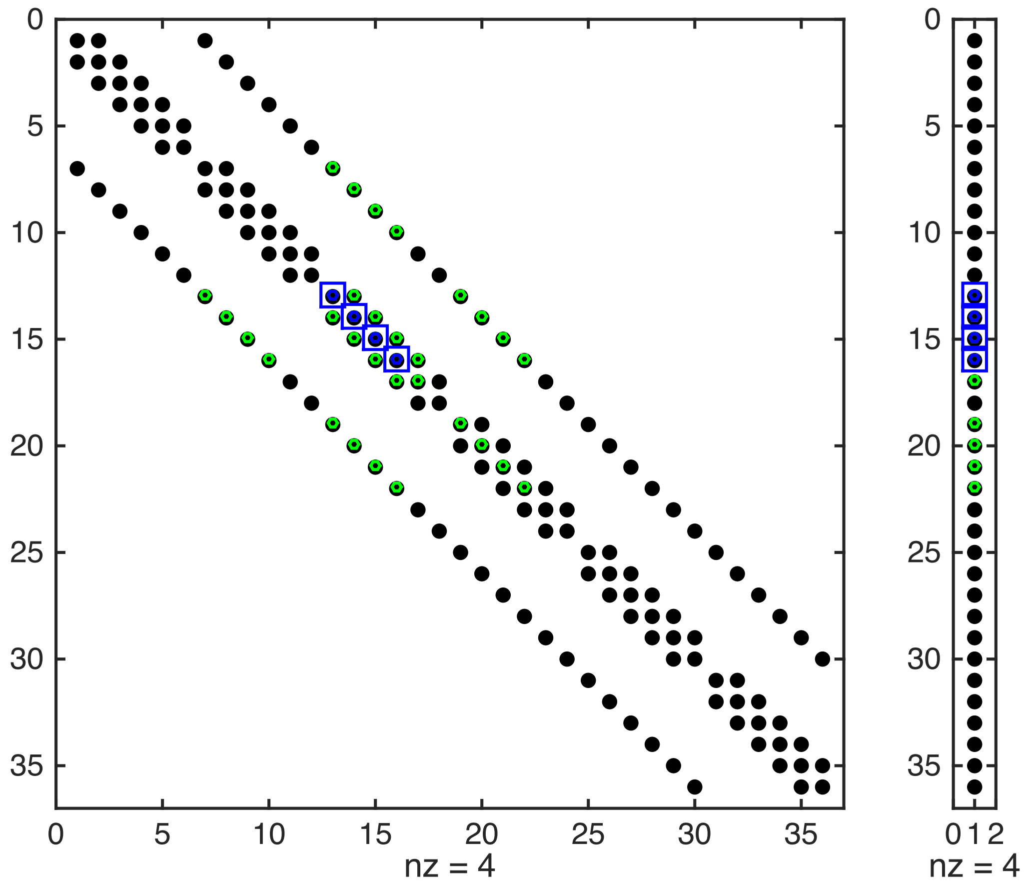
Normalized Iters L0: 1.00

Normalized Iters L1: 6.68

- **Overlap** improves convergence.
- **Random subdomain scheduling** (dashed lines) results in **slower** average **convergence**.

Directed Subdomain Extension

- **Overlap only in one direction, e.g. Top-Down overlap:**



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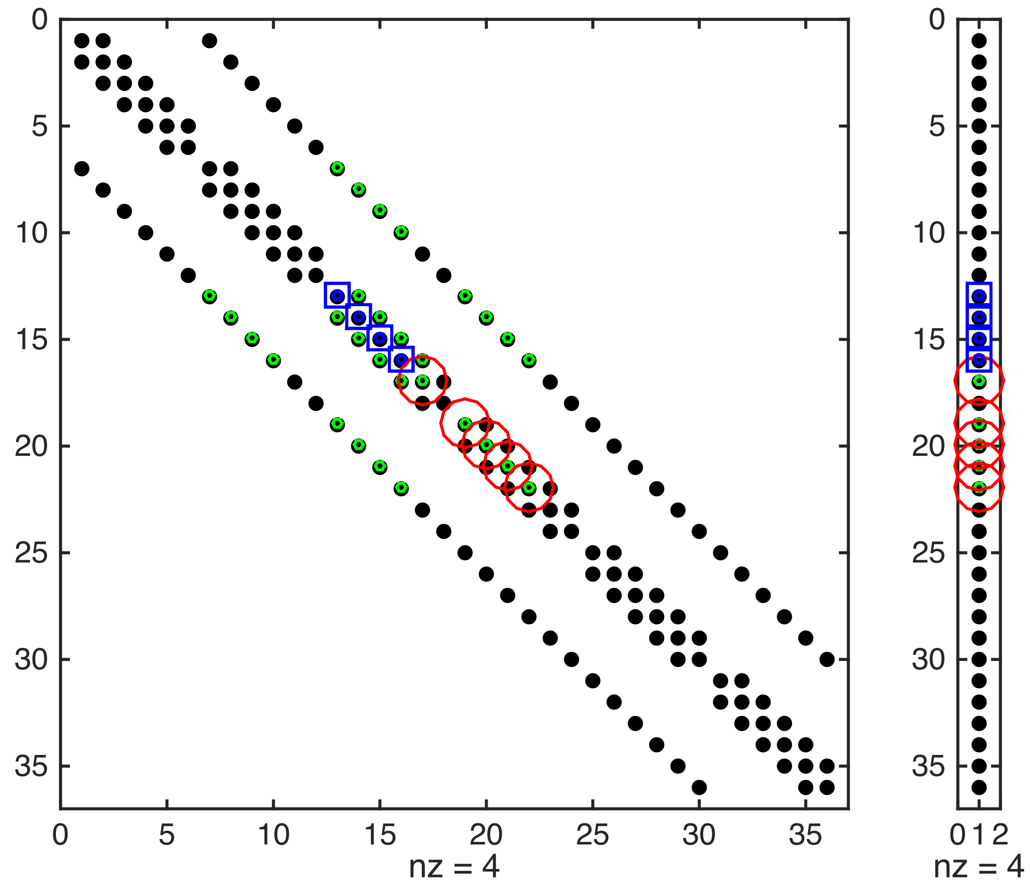

L-1 overlap

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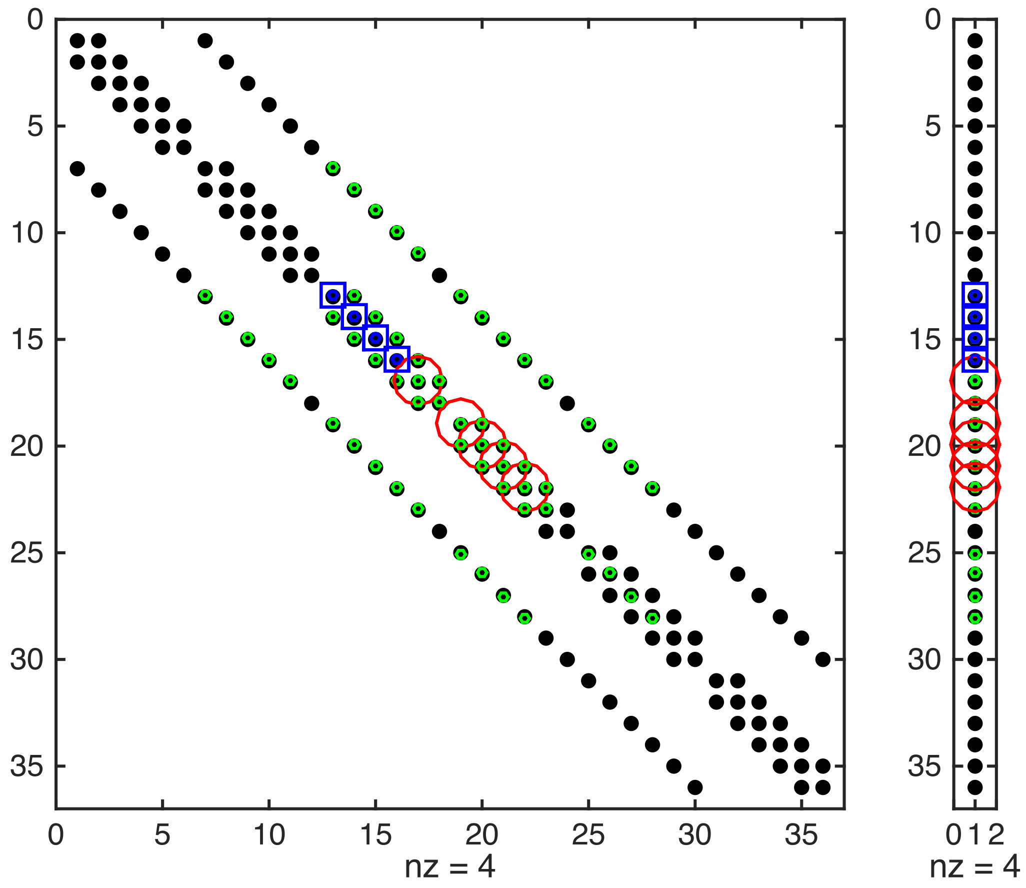

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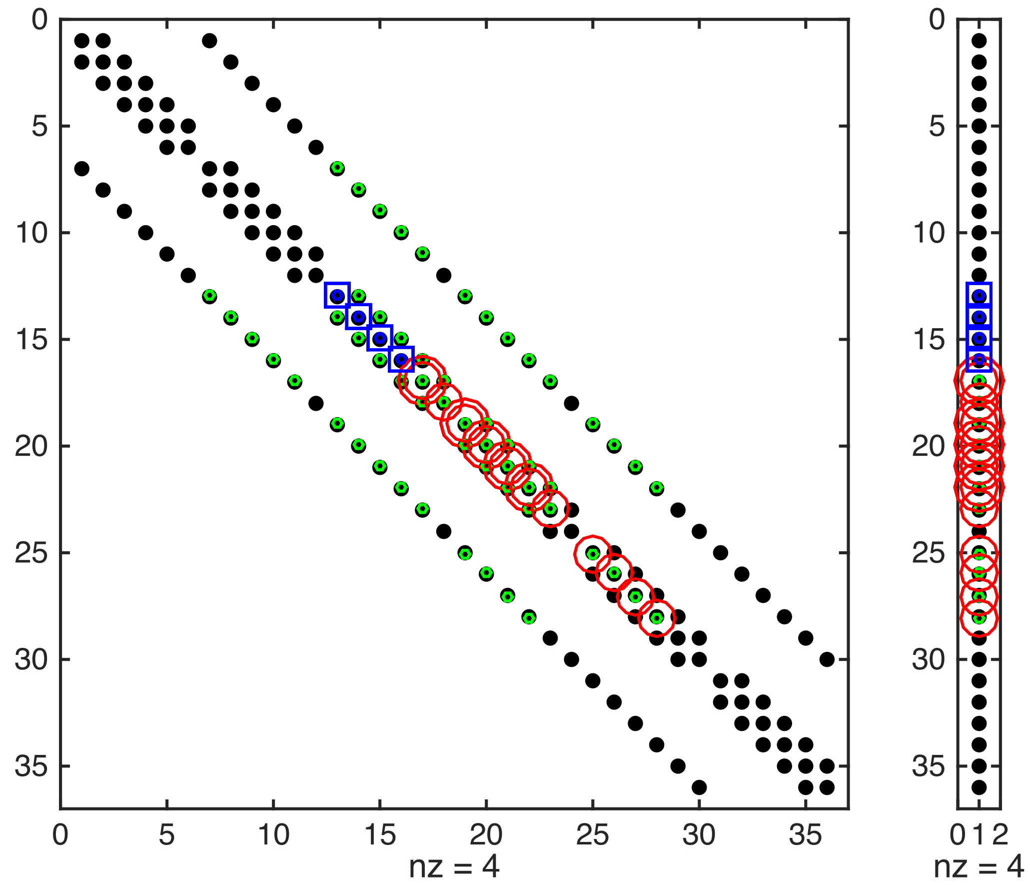
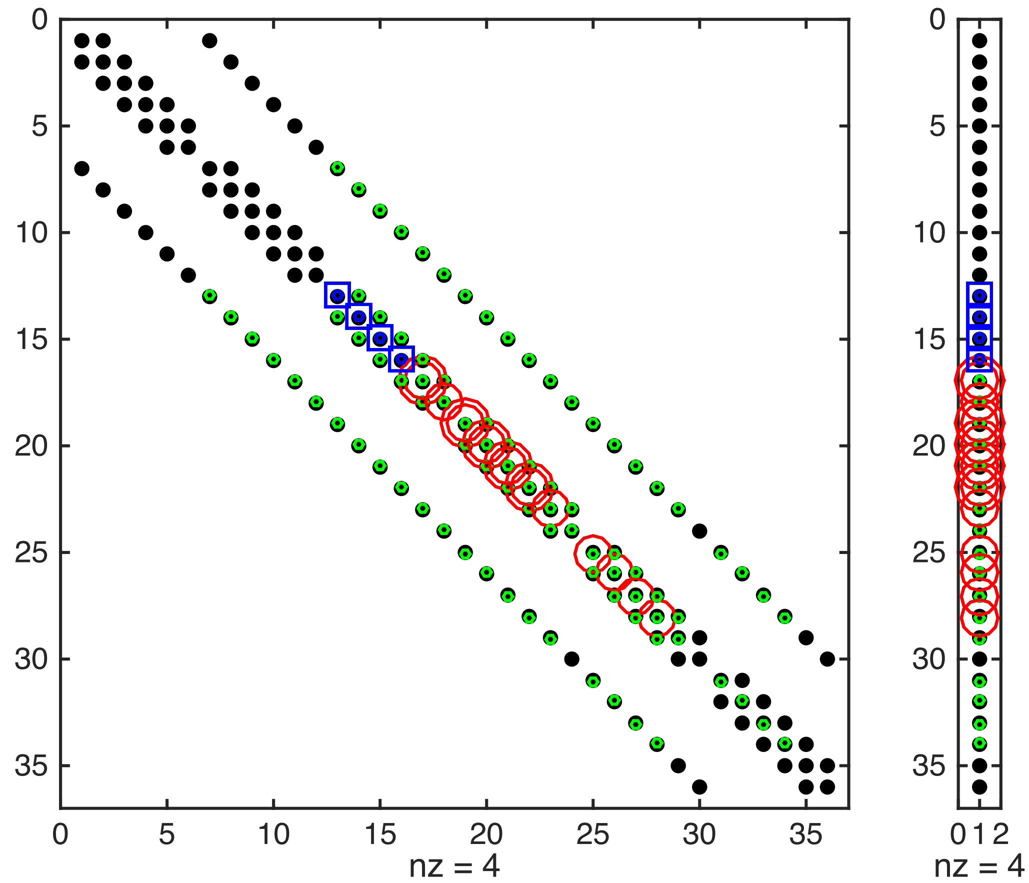


L-2 overlap

Directed Subdomain Extension

- **Overlap only in one direction, e.g. Top-Down overlap:**



Directed Subdomain Extension

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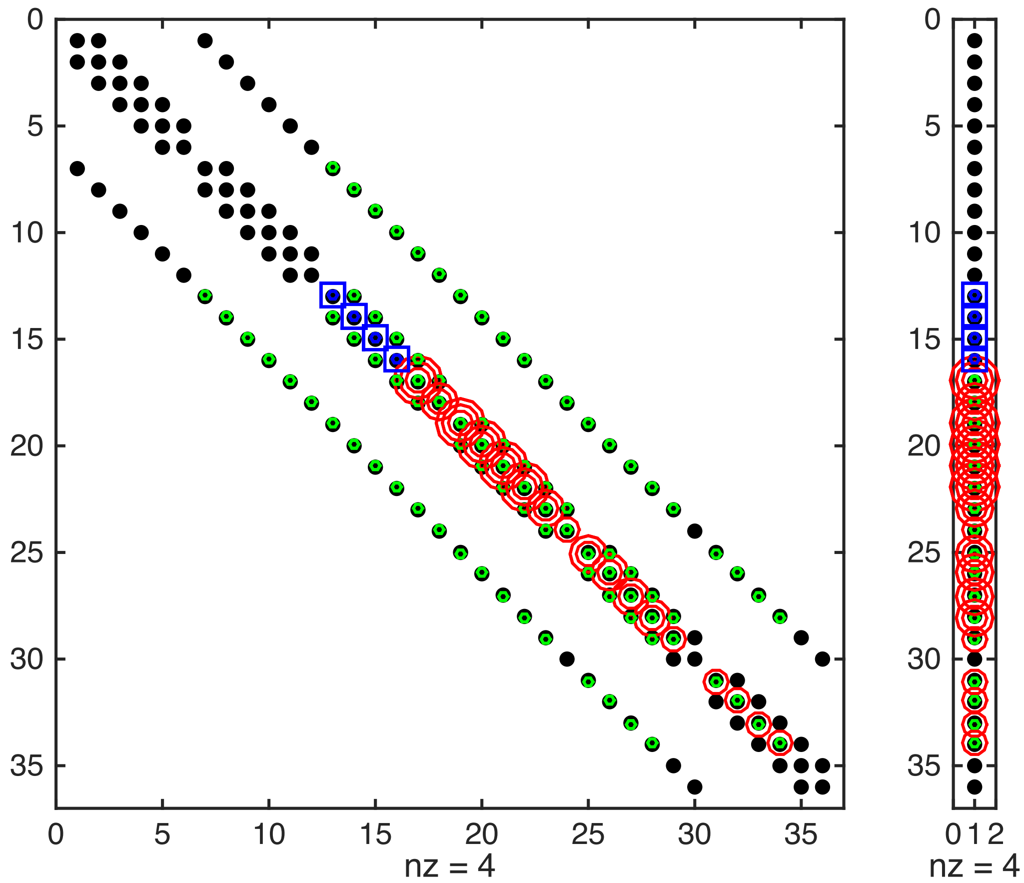
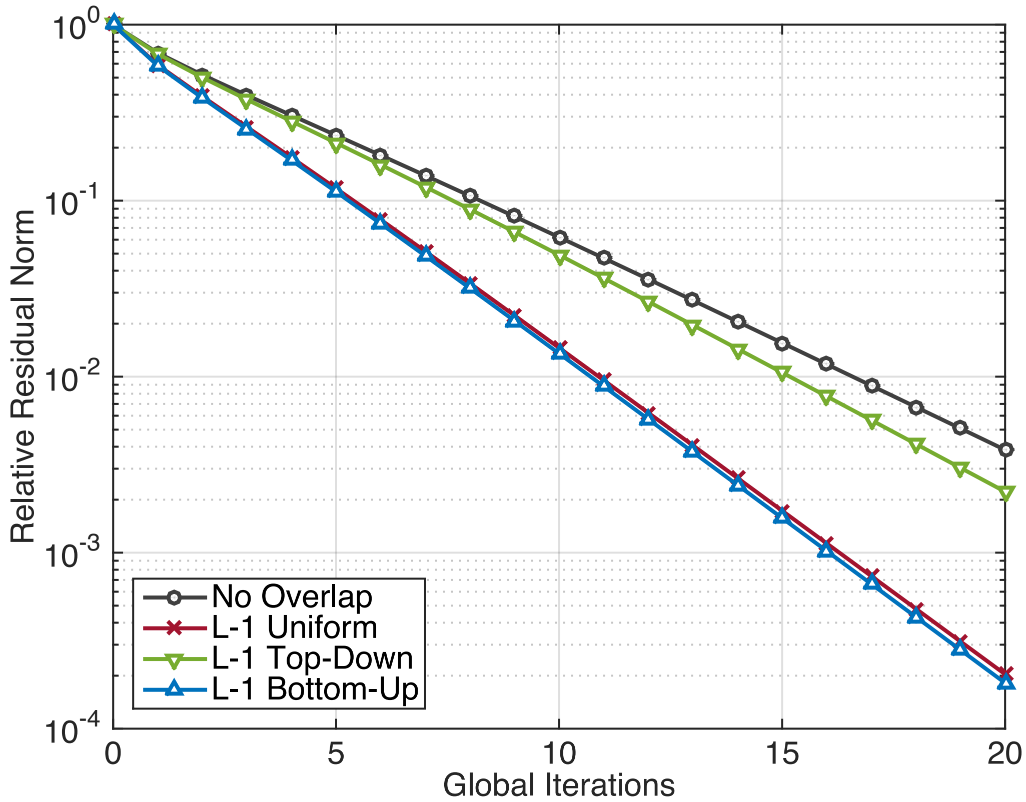


L-3 overlap

Subdomain only grows in one direction!

Directed Subdomain Extension

Test case: L3D 27-pt stencil



Normalized Iters L0: 1.00

Normalized Iters L1: 6.68

Normalized Iters L1: 3.83

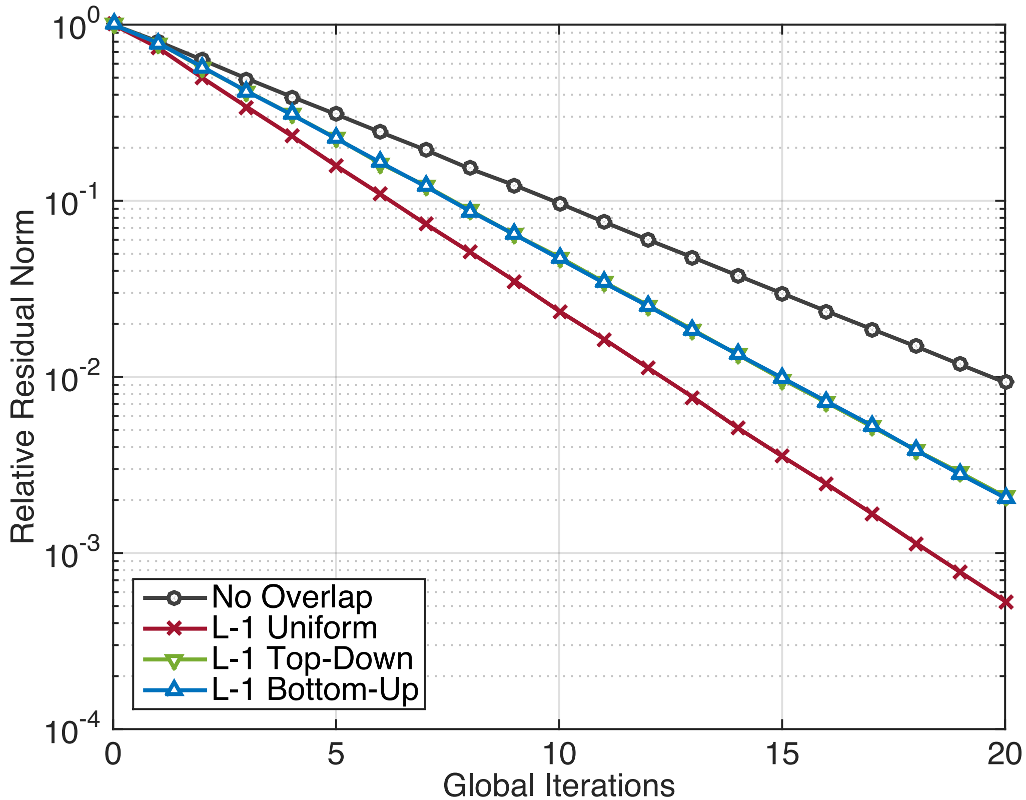
Normalized Iters L1: 3.83

For top-down subdomain scheduling:

- Bottom-Up overlap propagates all new information available in the uniform extension - at lower computational cost.
- Top-Down overlap provides almost no convergence benefit.

Directed Subdomain Extension

Test case: L3D 27-pt stencil



Normalized Iters L0: 1.00
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For random subdomain scheduling:

- No difference between Top-Down and Bottom-Up overlap.
- Symmetric matrix properties in combination with random update scheduling removes advantage of non-uniform extensions.

Sparse Triangular Systems

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  - Faster convergence expected for the scheduling:

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Sparse Triangular Systems

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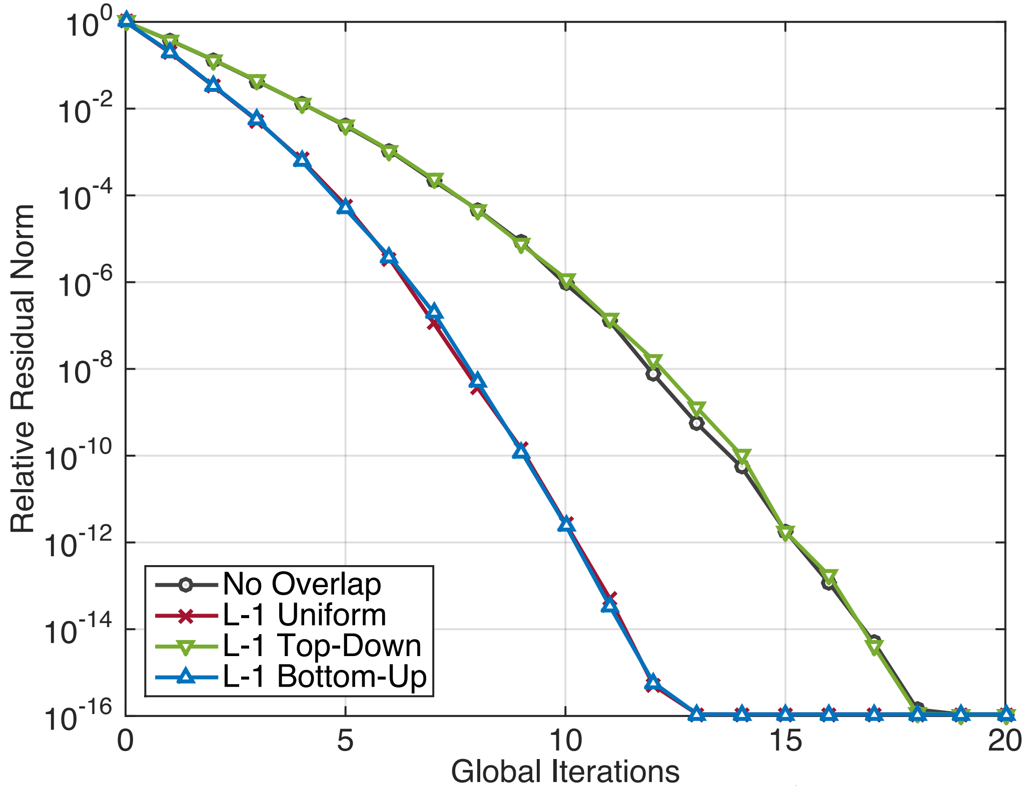
  - **Propagation** of **new information** in dependency direction is key.
  - Faster convergence expected for the scheduling:
    top-down in **Ly=b** and bottom-up in **Ux=y**.

- **Random subdomain scheduling**
  - No information on subdomain scheduling**.**
  - **Overlap useful** if it propagates **new information**.
  - **Directed subdomain extension opposite dependency direction**.

Directed Subdomain Extension

Test case: ILU(0) for L3D 27-pt stencil

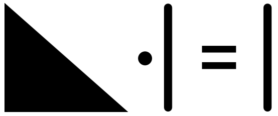


Normalized Iters L0: 1.00

Normalized Iters L1: 6.68

Normalized Iters L1: 3.83

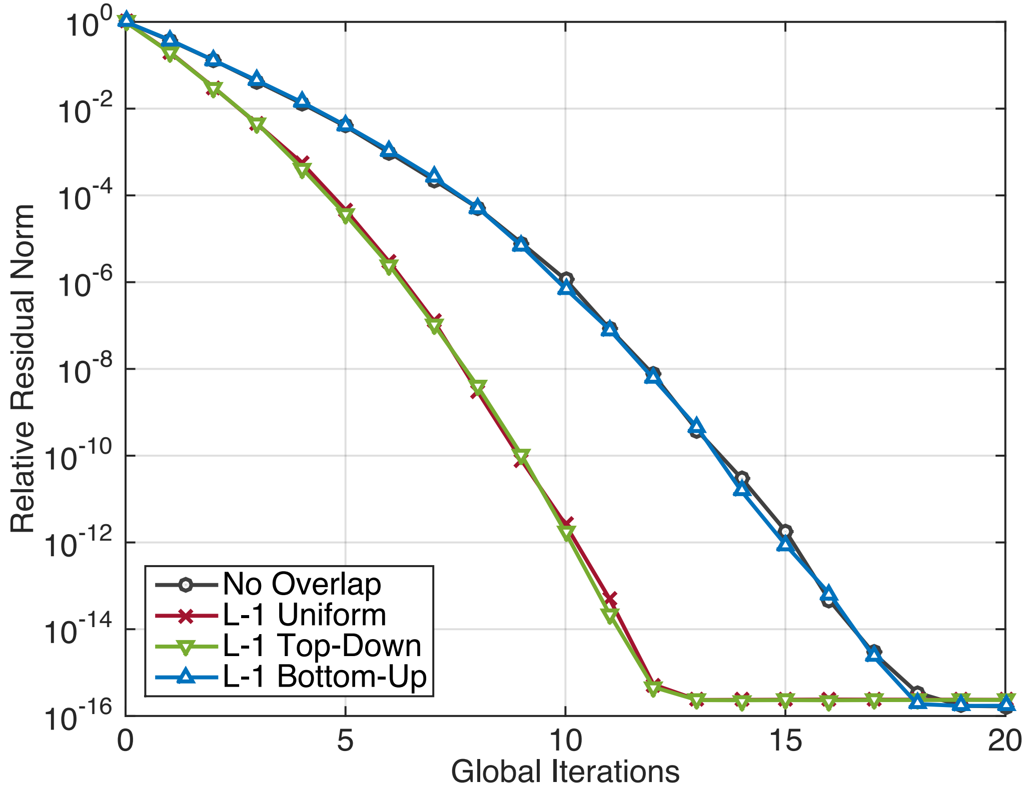
Normalized Iters L1: 3.83

For lower triangular system: 

- L-1 Bottom-Up overlap propagates all new information available in the uniform extension (left) - at lower computational cost (right).
- L-1 Top-Down overlap useless due to dependency in linear system.

Directed Subdomain Extension

Test case: ILU(0) for L3D 27-pt stencil

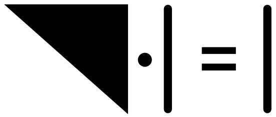


Normalized Iters L0: 1.00

Normalized Iters L1: 6.68

Normalized Iters L1: 3.83

Normalized Iters L1: 3.83

For upper triangular system: 

- L-1 Top-Down overlap propagates all new information available in the uniform extension (left) - at lower computational cost (right).
- L-1 Bottom-Up overlap useless due to dependency in linear system.

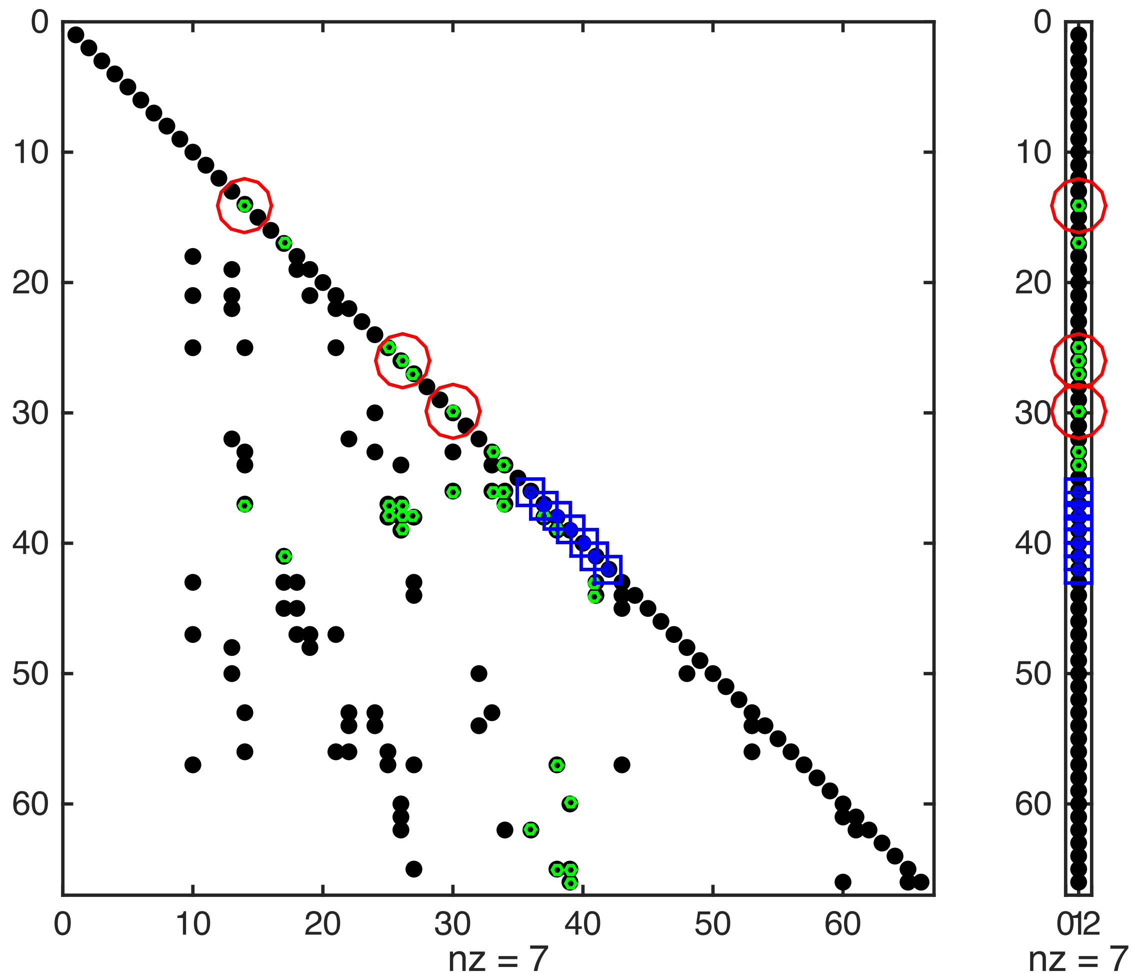
Anisotropic Problems

- **Anisotropic Fluid Flow**
  - E.g. groundwater flow.
  - Non-symmetric convection.
  - Information gets propagated faster in one than another direction.
  - Represented by different magnitude of matrix entries connecting unknowns.

- **Non-Uniform overlap accounting for Anisotropy**
  - Subdomain extended only by some candidates (largest matrix entries).
  - Recursive application of this strategy potentially results in subdomains that are different to any uniform extension.

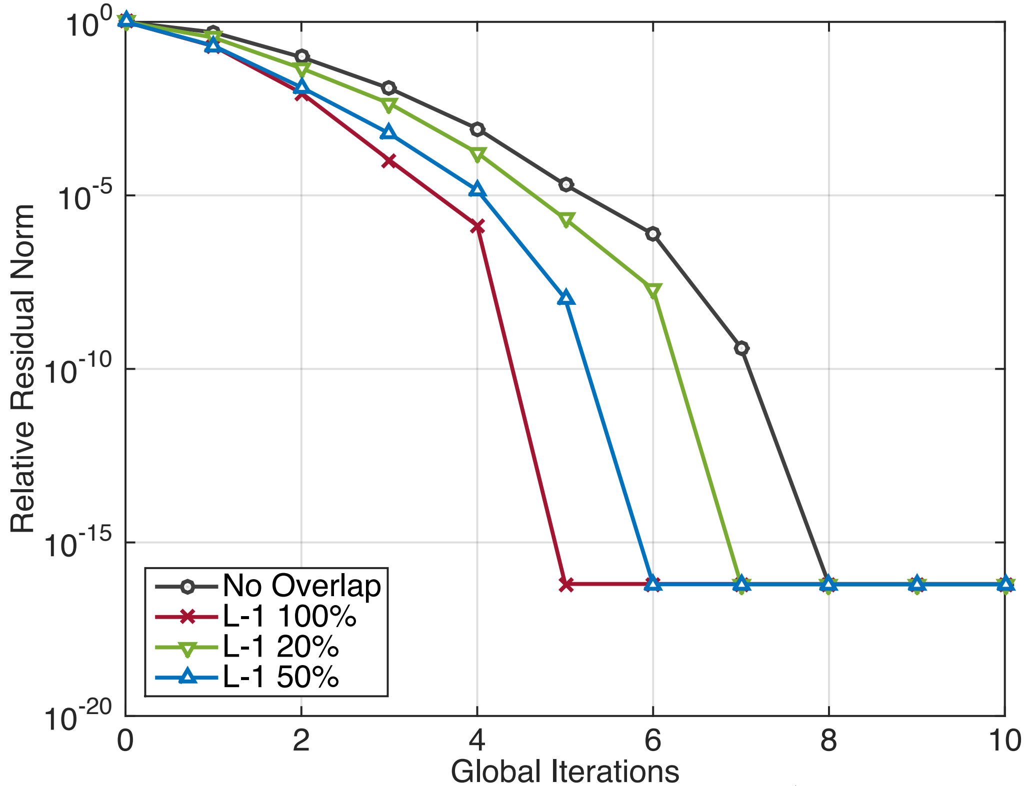
Anisotropic Problems

- Sparse triangular systems of ILU(0) for anisotropic fluid flow problem.
- Directed overlap opposite propagation direction.
- Recursive domain extension with only with some of the candidates.



Anisotropic Problems

Test case: ILU(0) for anisotropic problem



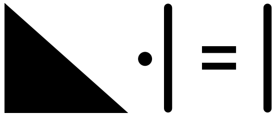
Normalized Iters L0: 1.00

Normalized Iters L1: 2.38

Normalized Iters L1: 1.33

Normalized Iters L1: 1.83

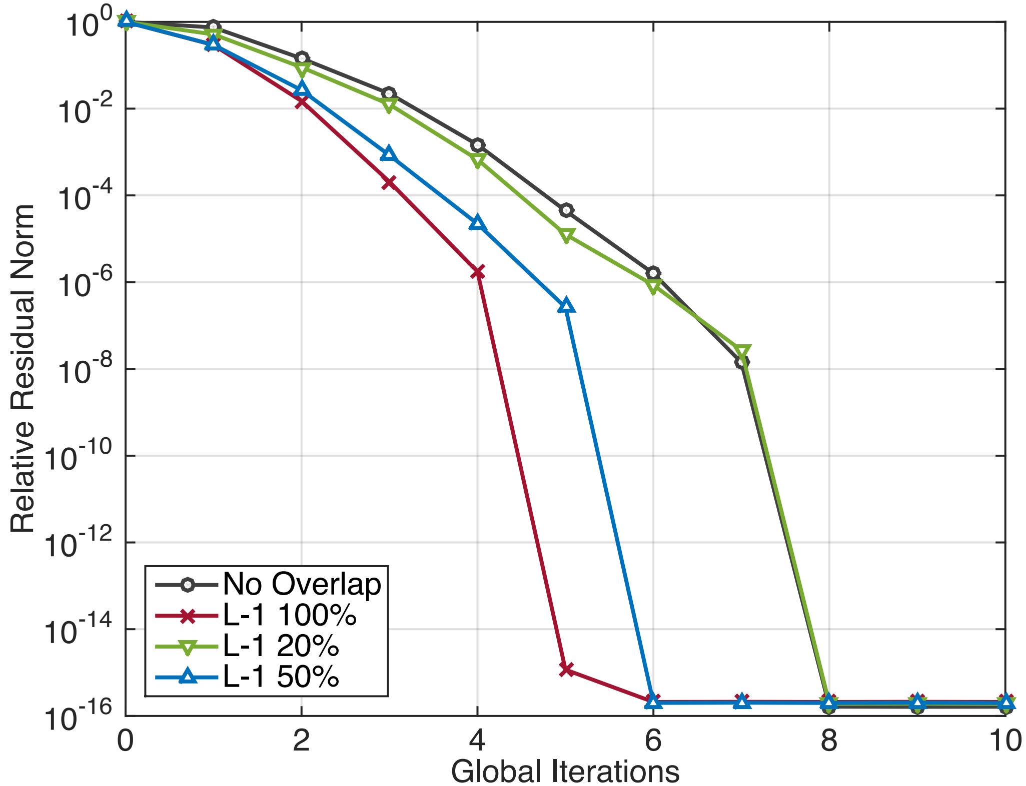
For lower triangular system:



- L-1 50% propagates all new information fast (left) and at low computational cost (right). (All Bottom-Up overlap.)
- L-1 20% is computationally cheaper, needs more global iterations.

Anisotropic Problems

Test case: ILU(0) for anisotropic problem



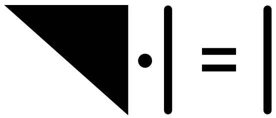
Normalized Iters L0: 1.00

Normalized Iters L1: 2.81

Normalized Iters L1: 1.35

Normalized Iters L1: 1.93

For upper triangular system:



- L-1 50% / L-1 80% matches L-1 100% in global iterations. (All Top-Down overlap.)
- L-1 50% computationally cheaper than L-1 100%.

Conclusion

- **Directed** overlap propagates information in a **certain direction**.
- Directed overlap **opposite** subdomain **scheduling** direction propagates only **new information**.
- For **triangular systems**, directed overlap **opposite dependency** works also for **random** subdomain **scheduling**.
- **Non-Uniform overlap** accounting for **anisotropy** propagates most **important information**.

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U.S. DEPARTMENT OF ENERGY

Normalized Iters Iterations

Ω_i Subdomains without overlap

$\bar{\Omega}_i$ Subdomains with overlap

$FP\left(\Omega_i\right)$ Floating point operations in local solver for subdomain

k Iterations

Normalized Iters Iterations:

$$\tilde{k} = k \cdot \frac{\sum_i FP\left(\bar{\Omega}_i\right)}{\sum_i FP\left(\Omega_i\right)}$$