CONDITION NUMBERS OF GAUSSIAN RANDOM MATRICES *

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Abstract. Let $G_{m\times n}$ be an $m \times n$ real random matrix whose elements are independent and identically distributed standard normal random variables, and let $\kappa_2(G_{m\times n})$ be the 2-norm condition number of $G_{m\times n}$. We prove that, for any $m \geq 2$, $n \geq 2$ and $x \geq |n-m|+1$, $\kappa_2(G_{m\times n})$ satisfies $\frac{1}{\sqrt{2\pi}}(c/x)^{|n-m|+1} < P\left(\frac{\kappa_2(G_{m\times n})}{n/(|n-m|+1)} > x\right) < \frac{1}{\sqrt{2\pi}}(C/x)^{|n-m|+1}$, where $0.245 \leq c \leq 2.000$ and $5.013 \leq C \leq 6.414$ are universal positive constants independent of m, n and x. Moreover, for any $m \geq 2$ and $n \geq 2$, $E(\log \kappa_2(G_{m\times n})) < \log \frac{n}{|n-m|+1} + 2.258$. A similar pair of results for complex Gaussian random matrices is also established.

Key words. Condition Number, Eigenvalues, Random Matrices, Singular Values, Wishart Distribution.

AMS subject classifications. 15A52, 15A12

1. Introduction.

In [5], Edelman obtained the limiting distributions and the limiting expected logarithms of the condition numbers of random rectangular matrices whose elements are independent and identically distributed standard normal random variables. The exact distributions of the condition numbers of $2 \times n$ matrices are also given in [5] by Edelman.

However, in the study of real-number and complex-number error correction codes based on random matrices [3] and their applications in fault tolerant high performance computing [4], in order to estimate the numerical stability and reliability of our coding schemes, we need to estimate the probabilities that the condition numbers of small random rectangular matrices are large. For example, what is the probability that the condition number of a 10×5 random matrix is larger than 10^2?

In this paper, we investigate the tails of the condition number distributions of random rectangular matrices whose elements are independent and identically distributed standard normal real or complex random variables. We establish upper and lower bounds for the tails of the condition number distributions of these matrices. Upper bounds for the expected logarithms of the condition numbers of these matrices are also given.

Based on our results, for random rectangular matrices whose elements are independent and identically distributed standard normal real or complex random variables, we are able to estimate the probabilities that their condition numbers are large. For example, based on our results, we are able to tell, for a 10×5 real random matrix whose elements are independent and identically distributed standard normal random variables, the probability that the condition number is larger than 10^2 is less than 6×10^{-7}.

Our main results for the 2-norm condition number κ of an $m \times n$ real random matrix whose elements are independent and identically distributed standard normal

* This research was supported in part by the Applied Mathematical Sciences Research Program of the Office of Mathematical, Information, and Computational Sciences, U.S. Department of Energy under contract DE-AC05-00OR22725 with UT-Battelle, LLC

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random variables are:

$$\frac{1}{\sqrt{2\pi}}\left(\frac{c}{x}\right)^{|n-m|+1} < P\left(\frac{\kappa}{n/(|n-m|+1)} > x\right) < \frac{1}{\sqrt{2\pi}}\left(\frac{C}{x}\right)^{|n-m|+1},$$

and

$$E(\log\kappa) < \log\frac{n}{|n-m|+1} + 2.258,$$

where $0.245 \le c \le 2.000$ and $5.013 \le C \le 6.414$ are universal positive constants independent of m, n and x, and $m \ge 2$, $n \ge 2$ and $x \ge |n-m|+1$.

For an $m \times n$ complex random matrix whose elements are independent and identically distributed standard normal random variables, our main results for the 2-norm condition number κ are:

$$\frac{1}{2\pi}\left(\frac{c}{x}\right)^{2(|n-m|+1)} < P\left(\frac{\kappa}{n/(|n-m|+1)} > x\right) < \frac{1}{2\pi}\left(\frac{C}{x}\right)^{2(|n-m|+1)},$$

and

$$E(\log\kappa) < \log\frac{n}{|n-m|+1} + 2.240,$$

where $0.319 \le c \le 2.000$ and $5.013 \le C \le 6.298$ are universal positive constants independent of m, n and x, and $m \ge 2$, $n \ge 2$ and $x \ge |n-m|+1$.

After finishing the manuscript of this paper, we communicated with Edelman and learned that similar problem was also being studied independently by Edelman and Sutton [7]. After simple formatting, the upper bounds in both papers actually can be unified into the same format

$$P(\kappa > x) \le C(m,n,\beta)\left(\frac{1}{x}\right)^{\beta(|n-m|+1)},$$

where $\beta = 1$ for real random matrices and $\beta = 2$ for complex random matrices, and $C(m,n,\beta)$ is a function of m, n, and β. However, the function $C(m,n,\beta)$ in the two papers do take very different forms and imply very different meanings.

On one hand, the bounds in [7] are asymptotically tight as $x \to \infty$ while the bounds in this paper are not. On the other hand, the bounds in this paper involve only elementary functions. Hence they are much simpler than the asymptotically tight bounds in [7] which involve high order moments of the largest eigenvalues of Wishart matrices. Although for the special case of large square random matrices, simple estimations for $C(m,n,\beta)$ are given in [7], for general rectangular matrices, no simple estimation is available.

It is well-known that the joint eigenvalue density function of a Wishart matrix has a closed form expression [9]. Therefore, $P(\kappa > x)$ can actually be expressed *accurately* as a high dimensional integration of this joint eigenvalue density function. One of the key aspects to estimate $P(\kappa > x)$ is to find a simple-to-use estimation of this accurate (but not simple-to-use) high dimensional integral expression. This paper is meaningful in that it finds out such a simple-to-use estimation by giving out simple upper and lower bounds which involve only elementary functions. We refer interested readers to [7] for more accurate asymptotically tight bounds and other related bounds for the tails of the condition numbers of general β-Laguerre ensembles.

Above and in what follows in this paper, the constant C and c denote universal positive constants independent of m, n and x; however, identical symbols may represent different numbers in different place.

2. Preliminaries and basic facts. Let X be an $m \times n$ matrix. If $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p$, where $p = \min\{m, n\}$, are the p singular values of X, then the *2-norm condition number* of X is

$$\kappa_2(X) = \frac{\sigma_1}{\sigma_p}.$$

For any $m \times n$ matrix X, X^T is an $n \times m$ matrix and $\kappa_2(X) = \kappa_2(X^T)$. So, without loss of generality, in discussing the condition numbers of random matrices, it is enough to only consider random matrices with no more rows than columns. Therefore, from now on, when we speak of an $m \times n$ matrix, we will assume $m \leq n$ in the rest of this paper.

Let $G_{m\times n}$ be an $m \times n$ real random matrix whose elements are independent and identically distributed standard normal random variables. Let $W_{m,n}$ denote the $m \times m$ random matrix $G_{m\times n}G^T_{m\times n}$. $W_{m,n}$ is the well known Wishart matrix named after John Wishart who has first studied its distribution.

Similar to [5], in this paper, we will study the condition number of $G_{m\times n}$ through investigating the eigenvalues of the Wishart matrix $W_{m,n}$. The following lemma establishes a simple relationship between the condition number of $G_{m\times n}$ and the eigenvalues of $W_{m,n}$.

PROPOSITION 2.1. *If λ_{max} is the largest eigenvalue of $W_{m,n}$, and λ_{min} is the smallest eigenvalue of $W_{m,n}$, then the 2-norm condition number of $G_{m\times n}$ satisfies*

$$\kappa_2(G_{m\times n}) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}}.$$

Remarkably enough, the exact joint probability density function for the m eigenvalues of the Wishart matrix $W_{m,n}$ can be written down in a closed form [9].

LEMMA 2.2. *If $\lambda_1 \geq ... \geq \lambda_m$ are the m eigenvalues of $W_{m,n}$, then the joint probability density function of $\lambda_1 \geq ... \geq \lambda_m$ is*

$$f(x_1, ..., x_m) = K_{m,n} e^{-\frac{1}{2}\sum_{i=1}^m x_i} \prod_{i=1}^m x_i^{\frac{1}{2}(n-m-1)} \prod_{i=1}^{m-1} \prod_{j=i+1}^m (x_i - x_j), \tag{2.1}$$

where

$$K^{-1}_{m,n} = \left(\frac{2^n}{\pi}\right)^{m/2} \prod_{i=1}^m \Gamma\left(\frac{n-m+i}{2}\right) \Gamma\left(\frac{i}{2}\right). \tag{2.2}$$

Let $N(0, 1)$ denote the standard normal distribution. Let $\widetilde{N}(0, 1)$ denote the distribution of $u + iv$, where u and v are independent and identically distributed $N(0, 1)$ random variables, and $i = \sqrt{-1}$. Let $\widetilde{G}_{m\times n}$ be an $m \times n$ complex random matrix whose elements are independent and identically distributed $\widetilde{N}(0, 1)$ random variables. Let $\widetilde{W}_{m,n}$ denote the $m \times m$ random matrix $\widetilde{G}_{m\times n}\widetilde{G}^H_{m\times n}$. In literature, $\widetilde{W}_{m,n}$ is called the complex Wishart matrix.

Similar to the real case, there is also a simple relationship between the condition number of $\widetilde{G}_{m\times n}$ and the eigenvalues of $\widetilde{W}_{m,n}$.

PROPOSITION 2.3. *If $\widetilde{\lambda}_{max}$ is the largest eigenvalue of $\widetilde{W}_{m,n}$, and $\widetilde{\lambda}_{min}$ is the smallest eigenvalue of $\widetilde{W}_{m,n}$, then the 2-norm condition number of $\widetilde{G}_{m\times n}$ satisfies*

$$\kappa_2(\widetilde{G}_{m\times n}) = \sqrt{\frac{\widetilde{\lambda}_{max}}{\widetilde{\lambda}_{min}}}.$$

Like the real case, the exact joint probability density function for the m eigenvalues of the complex Wishart matrix $\widetilde{W}_{m,n}$ can also be written down in a closed form [9].

LEMMA 2.4. *If $\widetilde{\lambda}_1 \geq ... \geq \widetilde{\lambda}_m$ are the m eigenvalues of $\widetilde{W}_{m,n}$, then the joint probability density function of $\widetilde{\lambda}_1 \geq ... \geq \widetilde{\lambda}_m$ is*

$$\widetilde{f}(x_1, ..., x_m) = \widetilde{K}_{m,n} e^{-\frac{1}{2}\sum_{i=1}^m x_i} \prod_{i=1}^{m} x_i^{n-m} \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (x_i - x_j)^2, \tag{2.3}$$

where

$$\widetilde{K}_{m,n}^{-1} = 2^{mn} \prod_{i=1}^{m} \Gamma\left(n-m+i\right) \Gamma\left(i\right). \tag{2.4}$$

In the process of deriving our upper and lower bounds for the tails of the condition number distributions, some bounds for Gamma and incomplete Gamma functions are very useful.

LEMMA 2.5. *Assume $a > 0$, and $b > 0$. If $t \leq \frac{b}{a}$, then*

$$\int_0^t e^{-ax} x^b dx \leq e^{-at} t^{b+1}.$$

Proof. Let $f(t) = \int_0^t e^{-ax} x^b dx - e^{-at} t^{b+1}$, then $f'(t) = e^{-at} t^b (1 + at - (b+1))$. So $f(t)$ decreases on $[0, \frac{b}{a}]$ and increases on $[\frac{b}{a}, \infty)$. Since $f(0) = 0$, and $f(\infty) = \int_0^\infty e^{-ax} x^b dx > 0$, if $t \leq \frac{b}{a}$, then $f(t) < 0$. Therefore, if $t \leq \frac{b}{a}$, then $\int_0^t e^{-ax} x^b dx \leq e^{-at} t^{b+1}$. ◻

LEMMA 2.6. *Assume $a > 0$, $b > 0$, and $k > \frac{1}{a}$. If $t \geq \frac{kb}{ka-1}$, then*

$$\int_t^\infty e^{-ax} x^b dx \leq k e^{-at} t^b.$$

Proof. Let $f(t) = \int_t^\infty e^{-ax} x^b dx - k e^{-at} t^b$, then $f'(t) = e^{-at} t^b (-1 + ka - \frac{kb}{t})$. So $f(t)$ decreases on $[0, \frac{kb}{ka-1}]$ and increases on $[\frac{kb}{ka-1}, \infty)$. Since $f(0) = \int_0^\infty e^{-ax} x^b dx > 0$, and $f(\infty) = 0$. So, if $t \geq \frac{kb}{ka-1}$, then $f(t) < 0$. Therefore, if $t \leq \frac{kb}{ka-1}$, then $\int_t^\infty e^{-ax} x^b dx \leq k e^{-at} t^b$. ◻

LEMMA 2.7. *If $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, where $x > 0$, then*

$$\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} < \Gamma(x+1) < \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x+\frac{1}{12x}}, \tag{2.5}$$

and

$$\Gamma(x + \frac{1}{2}) < \Gamma(x)\sqrt{x}. \tag{2.6}$$

Proof. (2.5) follows straightforwardly from 6.1.38 in [1], and (2.6) can be obtained from answer to Problem 9.60 in [8]. ◻

3. Bounds for eigenvalue densities of Wishart matrices. In this section, we will prove some bounds for the probability density functions of the eigenvalues of Wishart matrices. These bounds are very useful in the derivation of the bounds for the tails of the condition number distributions.

Let λ_{max} denote the largest eigenvalue of $W_{m,n}$, and λ_{min} denote the smallest eigenvalue of $W_{m,n}$. In the following lemma, we prove an upper bound for the joint probability density function of λ_{max} and λ_{min}.

LEMMA 3.1. *Let $f_{\lambda_{max},\lambda_{min}}(x,y)$ denote the joint probability density function of λ_{max} and λ_{min}, then $f_{\lambda_{max},\lambda_{min}}(x,y)$ satisfies:*

$$f_{\lambda_{max},\lambda_{min}}(x,y) \le C_{m,n} e^{-\frac{1}{2}(x+y)} x^{\frac{1}{2}(n+m-3)} y^{\frac{1}{2}(n-m-1)}, \tag{3.1}$$

where

$$C_{m,n} = \frac{1}{4\Gamma(m-1)\Gamma(n-m+1)}. \tag{3.2}$$

Proof. Let $R_{x,y} = \{(x_2, x_3, ..., x_{m-1}) : x \ge x_2 \ge ... \ge x_{m-1} \ge y\} \subseteq R^{m-2}$. From the joint probability density function of the m eigenvalues of $W_{m,n}$ in Lemma 2.2, we have

$$\begin{aligned} f_{\lambda_{max},\lambda_{min}}(x,y) &= \int_{R_{x,y}} f(x, x_2, ..., x_{m-1}, y) dx_2 dx_3 ... dx_{m-1} \\ &= K_{m,n} e^{-\frac{1}{2}(x+y)} x^{\frac{1}{2}(n-m-1)} y^{\frac{1}{2}(n-m-1)} \\ &\quad \int_{R_{x,y}} e^{-\frac{1}{2}\sum_{i=2}^{m-1} x_i} \prod_{i=2}^{m-1} x_i^{\frac{1}{2}(n-m-1)} \\ &\quad (x-y) \prod_{i=2}^{m-1} (x-x_i)(x_i-y) \prod_{i=2}^{m-2} \prod_{j=i+1}^{m-1} (x_i - x_j) \prod_{i=2}^{m-1} dx_i. \end{aligned} \tag{3.3}$$

Let $R_{m-2} = \{(x_2, x_3, ..., x_{m-1}) : x_2 \ge ... \ge x_{m-1} \ge 0\}$, then $R_{m-2} \subseteq R_{x,y}$. Note that, in (3.3), $x \ge x_i \ge y$ for $i = 2, 3, ..., m-1$. Replacing $x - y$ and $x - x_i$ by x, and $x_i - y$ by x_i for $i = 2, 3, ..., m-1$, and $R_{x,y}$ by R_{m-2}, then we get

$$\begin{aligned} f_{\lambda_{max},\lambda_{min}}(x,y) &\le K_{m,n} e^{-\frac{1}{2}(x+y)} x^{\frac{1}{2}(n+m-3)} y^{\frac{1}{2}(n-m-1)} \\ &\quad \int_{R_{m-2}} e^{-\frac{1}{2}\sum_{i=2}^{m-1} x_i} \prod_{i=2}^{m-1} x_i^{\frac{1}{2}(n-m+1)} \prod_{i=2}^{m-2} \prod_{j=i+1}^{m-1} (x_i - x_j) \prod_{i=2}^{m-1} dx_i. \end{aligned} \tag{3.4}$$

Note that $f(x_1, x_2, ..., x_m)$ in (2.1) is a probability density function, therefore, for any $m \le n$, we have

$$\int_{R_m} e^{-\frac{1}{2}\sum_{i=1}^{m} x_i} \prod_{i=1}^{m} x_i^{\frac{1}{2}(n-m-1)} \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (x_i - x_j) \prod_{i=1}^{m} dx_i = K_{m,n}^{-1},$$

where $R_m = \{x_1 \ge x_2 \ge ... \ge x_m \ge 0\} \subseteq R^m$. Therefore, we have

$$\int_{R_{m-2}} e^{-\frac{1}{2}\sum_{i=2}^{m-1} x_i} \prod_{i=2}^{m-1} x_i^{\frac{1}{2}(n-m+1)} \prod_{i=2}^{m-2} \prod_{j=i+1}^{m-1} (x_i - x_j) \prod_{i=2}^{m-1} dx_i = K_{m-2,n}^{-1}. \tag{3.5}$$

Substitute (3.5) into (3.4), we obtain

$$f_{\lambda_{max},\lambda_{min}}(x,y) \leq \frac{K_{m,n}}{K_{m-2,n}} e^{-\frac{1}{2}(x+y)} x^{\frac{1}{2}(n+m-3)} y^{\frac{1}{2}(n-m-1)}. \tag{3.6}$$

From (2.2), we have

$$\begin{aligned} \frac{K_{m,n}}{K_{m-2,n}} &= \frac{\pi}{2^n} \frac{1}{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)} \\ &= \frac{1}{4\Gamma\left(m-1\right)\Gamma\left(n-m+1\right)}. \end{aligned} \tag{3.7}$$

Substitute (3.6) into (3.5), we get (3.1) and (3.2) . ∎

Let $\widetilde{\lambda}_{max}$ denote the largest eigenvalue of $\widetilde{W}_{m,n}$, and $\widetilde{\lambda}_{min}$ denote the smallest eigenvalue of $\widetilde{W}_{m,n}$. Similar to the real case, in the following lemma, we give an upper bound for the joint probability density function of $\widetilde{\lambda}_{max}$ and $\widetilde{\lambda}_{min}$. The upper bound in complex case can be proved using the same techniques used in the real case. Therefore, we omit the proof and only give the result here.

Lemma 3.2. *Let $\widetilde{f}_{\widetilde{\lambda}_{max},\widetilde{\lambda}_{min}}(x,y)$ denote the joint probability density function of $\widetilde{\lambda}_{max}$ and $\widetilde{\lambda}_{min}$, then $\widetilde{f}_{\widetilde{\lambda}_{max},\widetilde{\lambda}_{min}}(x,y)$ satisfies:*

$$\widetilde{f}_{\widetilde{\lambda}_{max},\widetilde{\lambda}_{min}}(x,y) \leq \widetilde{C}_{m,n} e^{-\frac{1}{2}(x+y)} x^{n+m-2} y^{n-m}, \tag{3.8}$$

where

$$\widetilde{C}_{m,n} = \frac{1}{2^{2n}\Gamma(m-1)\Gamma(m)\Gamma(n-m+1)\Gamma(n-m+2)}. \tag{3.9}$$

Bounds for the probability density functions of the smallest eigenvalues are also very useful in the derivation of the bounds for the tails of the condition number distributions. In the following lemma, we prove upper and lower bounds for the probability density function of the smallest eigenvalue of a real Wishart matrix.

Lemma 3.3. *Let $f_{\lambda_{min}}(x)$ denotes the probability density function of the smallest eigenvalue of $W_{m,n}$, then $f_{\lambda_{min}}(x)$ satisfies:*

$$L_{m,n} e^{-\frac{m}{2}x} x^{\frac{1}{2}(n-m-1)} \leq f_{\lambda_{min}}(x) \leq L_{m,n} e^{-\frac{1}{2}x} x^{\frac{1}{2}(n-m-1)}, \tag{3.10}$$

where

$$L_{m,n} = \frac{2^{\frac{n-m-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(n-m+1\right)}. \tag{3.11}$$

Proof. Let $R_x = \{(x_1, x_2, ..., x_{m-1}) : x_1 \geq ... \geq x_{m-1} \geq x\} \subseteq R^{m-1}$. From the joint probability density function of the eigenvalues of $W_{m,n}$ in Lemma 2.2, we have

$$\begin{aligned} f_{\lambda_{min}}(x) &= \int_{R_x} f(x_1, x_2, ..., x_{m-1}, x) dx_1 dx_2 dx_{m-1} \\ &= K_{m,n} e^{-\frac{1}{2}x} x^{\frac{1}{2}(n-m-1)} \int_{R_x} e^{-\frac{1}{2}\sum_{i=1}^{m-1} x_i} \prod_{i=1}^{m-1} x_i^{\frac{1}{2}(n-m-1)} \\ &\quad \prod_{i=1}^{m-1}(x_i - x) \prod_{i=1}^{m-2}\prod_{j=i+1}^{m-1}(x_i - x_j) \prod_{i=1}^{m-1} dx_i. \end{aligned}$$

For the lower bound part, taking the transformation $y_i = x_i - x$, where $i = 1, 2, ..., m-1$, we have

$$f_{\lambda_{min}}(x) = K_{m,n} e^{-\frac{m}{2}x} x^{\frac{1}{2}(n-m-1)} \int_{R_y} e^{-\frac{1}{2}\sum_{i=1}^{m-1} y_i} \prod_{i=1}^{m-1} (y_i + x)^{\frac{1}{2}(n-m-1)} \prod_{i=1}^{m-1} y_i \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (y_i - y_j) \prod_{i=1}^{m-1} dy_i,$$

where $R_y = \{y_1 \geq y_2 \geq ... \geq y_{m-1} \geq 0\} \subseteq R^{m-1}$.

Replacing $y_i + x$ by y_i for $i = 1, 2, ..., m-1$, we obtain

$$f_{\lambda_{min}}(x) \geq K_{m,n} e^{-\frac{m}{2}x} x^{\frac{1}{2}(n-m-1)} \int_{R_y} e^{-\frac{1}{2}\sum_{i=1}^{m-1} y_i} \prod_{i=1}^{m-1} y_i^{\frac{1}{2}(n-m+1)} \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (y_i - y_j) \prod_{i=1}^{m-1} dy_i.$$

Note that

$$\int_{R_y} e^{-\frac{1}{2}\sum_{i=1}^{m-1} y_i} \prod_{i=1}^{m-1} y_i^{\frac{1}{2}(n-m+1)} \prod_{i=1}^{m-2} \prod_{j=i+1}^{m-1} (y_i - y_j) \prod_{i=1}^{m-1} dy_i = K_{m-1,n+1}^{-1}.$$

Therefore, we obtain

$$f_{\lambda_{min}}(x) \geq \frac{K_{m,n}}{K_{m-1,n+1}} e^{-\frac{m}{2}x} x^{\frac{1}{2}(n-m-1)}. \tag{3.12}$$

For the upper bound part, from [6], we have

$$f_{\lambda_{min}}(x) \leq \frac{K_{m,n}}{K_{m-1,n+1}} e^{-\frac{1}{2}x} x^{\frac{1}{2}(n-m-1)}. \tag{3.13}$$

From (2.2), we have

$$\begin{aligned} \frac{K_{m,n}}{K_{m-1,n+1}} &= \frac{\sqrt{\pi}\left(\frac{1}{2}\right)^{\frac{n-m+1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)} \\ &= \frac{2^{\frac{n-m-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(n-m+1\right)}. \end{aligned} \tag{3.14}$$

Substitute (3.14) into (3.13) and (3.12), we get (3.10) and (3.11) . ∎

Similar to the real case, in the following lemma, we give upper and lower bounds for the probability density function of the smallest eigenvalue $\widetilde{\lambda}_{min}$ of a complex Wishart matrix. These bounds can be proved using the same techniques used in the real case. Therefore, we omit the proof and only give the result here.

Lemma 3.4. *Let $\widetilde{f}_{\widetilde{\lambda}_{min}}(x)$ denotes the probability density function of the smallest eigenvalue of $\widetilde{W}_{m,n}$, then $\widetilde{f}_{\widetilde{\lambda}_{min}}(x)$ satisfies:*

$$\widetilde{L}_{m,n} e^{-\frac{m}{2}x} x^{n-m} \leq \widetilde{f}_{\widetilde{\lambda}_{min}}(x) \leq \widetilde{L}_{m,n} e^{-\frac{1}{2}x} x^{n-m}, \tag{3.15}$$

where

$$\widetilde{L}_{m,n} = \frac{\Gamma(n+1)}{2^{n-m+1}\Gamma(m)\Gamma(n-m+1)\Gamma(n-m+2)}. \tag{3.16}$$

4. The upper bounds for the distribution tails. In this section, we will derive the upper bounds for the tails of the condition number distributions of random rectangular matrices whose elements are independent and identically distributed standard normal random variables. Our main results are Theorem 4.5 for real random matrices, and Theorem 4.6 for complex random matrices.

LEMMA 4.1. *For any $A > 0$, $x > 0$, and $n \geq m \geq 2$, the largest eigenvalue λ_{max} and the smallest eigenvalue λ_{min} of $W_{m,n}$ satisfy*

$$P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} \leq \frac{A^2 n}{x^2}\right) < \frac{1}{\Gamma(n-m+2)}\left(\frac{An}{x}\right)^{n-m+1}.$$

Proof. From the upper bound for the probability density function of λ_{min} in Lemma 3.2, we have

$$\begin{aligned} P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} \leq \frac{A^2 n}{x^2}\right) &< P\left(\lambda_{min} \leq \frac{A^2 n}{x^2}\right) \\ &= \int_0^{\frac{A^2 n}{x^2}} f_{\lambda_{min}}(t)dt \\ &< L_{m,n} \int_0^{\frac{A^2 n}{x^2}} t^{\frac{1}{2}(n-m-1)} dt \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)^{\frac{n-m+1}{2}}} \frac{1}{\Gamma(n-m+2)}\left(\frac{An}{x}\right)^{n-m+1}. \end{aligned}$$

Since $m \leq n$, by applying (2.6) repeatedly, we can prove

$$\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)^{\frac{n-m+1}{2}} > \Gamma\left(\frac{n+1}{2}\right).$$

Therefore, we have

$$P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} \leq \frac{A^2 n}{x^2}\right) < \frac{1}{\Gamma(n-m+2)}\left(\frac{An}{x}\right)^{n-m+1}.$$

□

Similar to real random matrices, for complex random matrices, we have the following Lemma 4.2. Lemma 4.2 can be proved using the same techniques as Lemma 4.1, so we will omit the proof and only give the result.

LEMMA 4.2. *For any $A > 0$, $x > 0$, and $n \geq m \geq 2$, the largest eigenvalue $\widetilde{\lambda}_{max}$ and the smallest eigenvalue $\widetilde{\lambda}_{min}$ of $\widetilde{W}_{m,n}$ satisfy*

$$P\left(\frac{\widetilde{\lambda}_{max}}{\widetilde{\lambda}_{min}} > x^2, \widetilde{\lambda}_{min} \leq \frac{A^2 n}{x^2}\right) < \frac{1}{\Gamma(n-m+2)^2}\left(\frac{A^2 n^2}{2x^2}\right)^{n-m+1}.$$

The proof of the following Lemma 4.3 is based on the upper bound for the joint probability density function of λ_{max} and λ_{min} in Lemma 3.1 and the upper bound of the incomplete Gamma function in Lemma 2.6.

Lemma 4.3. *For any $A \geq 2.32$, $x > 0$, and $n \geq m \geq 2$, the largest eigenvalue λ_{max} and the smallest eigenvalue λ_{min} of $W_{m,n}$ satisfy*

$$P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) < 0.017 \frac{1}{\Gamma(n-m+2)} \left(\frac{An}{x}\right)^{n-m+1}.$$

Proof. From the upper bound for the joint probability density function of λ_{max} and λ_{min} in Lemma 3.1, we have

$$\begin{aligned} P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) &= \int_{\frac{A^2 n}{x^2}}^{\infty} \int_{tx^2}^{\infty} f_{\lambda_{max},\lambda_{min}}(s,t) ds dt \\ &< \int_{\frac{A^2 n}{x^2}}^{\infty} \int_{tx^2}^{\infty} C_{m,n} e^{-\frac{1}{2}t} t^{\frac{1}{2}(n-m-1)} e^{-\frac{1}{2}s} s^{\frac{1}{2}(n+m-3)} ds dt. \end{aligned}$$

Taking the transform $u = tx^2$, we have

$$\begin{aligned} P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) = C_{m,n} \left(\frac{1}{x}\right)^{n-m+1} & \int_{A^2 n}^{\infty} e^{-\frac{u}{2x^2}} u^{\frac{1}{2}(n-m-1)} \\ & \left(\int_u^{\infty} e^{-\frac{1}{2}s} s^{\frac{1}{2}(n+m-3)} ds\right) du. \end{aligned}$$

According to Lemma 2.6, with $k = 4$,if $u \geq 2(n+m-3)$, then

$$\int_u^{\infty} e^{-\frac{1}{2}s} s^{\frac{1}{2}(n+m-3)} ds \leq 4 e^{-\frac{1}{2}u} u^{\frac{1}{2}(n+m-3)}.$$

Since $A \geq 2.32$ and $n \geq m$, hence, $u \geq A^2 n \geq 2(n+m-3)$. Therefore, we have

$$\begin{aligned} P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) &\leq 4 C_{m,n} \left(\frac{1}{x}\right)^{n-m+1} \int_{A^2 n}^{\infty} e^{-\frac{u}{2x^2} - \frac{1}{2}u} u^{n-2} du \\ &\leq 4 C_{m,n} \left(\frac{1}{x}\right)^{n-m+1} \int_{A^2 n}^{\infty} e^{-\frac{1}{2}u} u^{n-2} du. \end{aligned}$$

Since $A \geq 2.32$, so $A^2 n \geq 4(n-2)$. Apply Lemma 2.6 again, we have

$$\begin{aligned} P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) &\leq 16 C_{m,n} e^{-\frac{1}{2}A^2 n} A^{2n-4} n^{n-2} \left(\frac{1}{x}\right)^{n-m+1} \\ &= \frac{4 e^{-\frac{A^2 n}{2}} A^{2n-4} n^{m-3}}{\Gamma(m-1)\Gamma(n-m+1)} \left(\frac{n}{x}\right)^{n-m+1} \\ &\leq \frac{4 e^{(2\ln A - \frac{A^2}{2})n}}{A^4} \frac{n^{m-2}}{\Gamma(m-1)} \frac{1}{\Gamma(n-m+2)} \left(\frac{n}{x}\right)^{n-m+1}. \end{aligned} \tag{4.1}$$

Note that, for any $2 \leq m \leq n$, it can be proved that

$$\frac{n^{m-2}}{\Gamma(m-1)} < \frac{e^n}{\sqrt{4\pi}}. \tag{4.2}$$

Substitute (4.2) into (4.1), we have

$$P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) \leq \frac{4 e^{(2\ln A - \frac{A^2}{2} + 1)n}}{\sqrt{4\pi} A^4} \frac{1}{\Gamma(n-m+2)} \left(\frac{n}{x}\right)^{n-m+1}.$$

Since $A \geq 2.32$, therefore, we have

$$e^{(2\ln A - \frac{A^2}{2}+1)n} < 1.$$

Therefore, when $A \geq 2.32$, we have

$$P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{A^2 n}{x^2}\right) \leq \frac{4}{\sqrt{4\pi}A^4}\frac{1}{\Gamma(n-m+2)}\left(\frac{n}{x}\right)^{n-m+1}$$
$$\leq 0.017\frac{1}{\Gamma(n-m+2)}\left(\frac{An}{x}\right)^{n-m+1}.$$

□

Similar to real random matrices, for complex random matrices, we have the following Lemma 4.4. Lemma 4.4 can be proved using the same techniques as Lemma 4.3, so we will omit the proof and only give the result.

LEMMA 4.4. *For any $A \geq 3.2735$, $x > 0$, and $n \geq m \geq 2$, the largest eigenvalue $\widetilde{\lambda}_{max}$ and the smallest eigenvalue $\widetilde{\lambda}_{min}$ of $\widetilde{W}_{m,n}$ satisfy*

$$P\left(\frac{\widetilde{\lambda}_{max}}{\widetilde{\lambda}_{min}} > x^2, \widetilde{\lambda}_{min} > \frac{A^2 n}{x^2}\right) < 0.0016\frac{1}{\Gamma(n-m+2)^2}\left(\frac{A^2 n^2}{2x}\right)^{n-m+1}.$$

We are now prepared to prove our first main result about the condition numbers of real random matrices whose elements are independent and identically distributed standard normal random variables.

THEOREM 4.5. *For any $n \geq m \geq 2$ and $x \geq n-m+1$, the 2-norm condition number of $G_{m\times n}$ satisfies*

$$P\left(\frac{\kappa_2(G_{m\times n})}{n/(n-m+1)} > x\right) < \frac{1}{\sqrt{2\pi}}\left(\frac{C}{x}\right)^{n-m+1}, \tag{4.3}$$

where $C \leq 6.414$ is a universal positive constant independent of m, n, and x.

Proof. For any $L > 0$, inspired by [2], we first break down $P(\kappa_2(G_{m\times n}) > x)$ into two parts.

$$P(\kappa_2(G_{m\times n}) > x) = P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2\right)$$
$$= P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} \leq \frac{L^2 n}{x^2}\right) + P\left(\frac{\lambda_{max}}{\lambda_{min}} > x^2, \lambda_{min} > \frac{L^2 n}{x^2}\right).$$

Let $L = 2.32$, then based on Lemma 4.1 and Lemma 4.3, we can get

$$P(\kappa_2(G_{m\times n}) > x) < \frac{1}{\Gamma(n-m+2)}\left(\frac{Ln}{x}\right)^{n-m+1}$$
$$+0.017\frac{1}{\Gamma(n-m+2)}\left(\frac{Ln}{x}\right)^{n-m+1}$$
$$< \frac{1}{\Gamma(n-m+2)}\left(\frac{1.017Ln}{x}\right)^{n-m+1}.$$

Note that, from Lemma 2.7, we have

$$\Gamma(n-m+2) > \sqrt{2\pi(n-m+1)}(n-m+1)^{n-m+1}e^{-(n-m+1)}.$$

Therefore, we have

$$P(\kappa_2(G_{m\times n}) > x) < \frac{1}{\sqrt{2\pi(n-m+1)}}\left(\frac{1.017eL\frac{n}{n-m+1}}{x}\right)^{n-m+1}.$$

Therefore

$$P\left(\frac{\kappa_2(G_{m\times n})}{n/(n-m+1)} > x\right) < \frac{1}{\sqrt{2\pi(n-m+1)}}\left(\frac{1.017eL}{x}\right)^{n-m+1} < \frac{1}{\sqrt{2\pi}}\left(\frac{6.414}{x}\right)^{n-m+1}.$$

Let $C = 6.414$, then we get (4.3). ▯

Remark:

1. The upper bound in Theorem 4.5 is for arbitrary $n \geq m \geq 2$ and $x \geq n-m+1$. For some special case of m and n, more precise upper bound can be obtained. For example, for the special case of real random $2 \times n$ matrices, based on the exact probability density function of $\kappa_2(G_{2\times n})$ in [5], we can get

$$P\left(\kappa_2(G_{2\times n}) > x\right) = \left(\frac{2x}{x^2+1}\right)^{n-1} < \left(\frac{2}{x}\right)^{n-1}.$$

2. For the special case of real random $m \times m$ matrices, where $m \geq 3$, it has been proved in [2] that

$$P\left(\kappa_2(G_{m\times m}) > m\,.\,x\right) < \frac{C'}{x}, \tag{4.4}$$

where $C' \leq 5.60$ is a universal positive constant independent of x and m.
In Theorem 4.5, if we take $m = n$, then we have

$$P\left(\kappa_2(G_{m\times m}) > m\,.\,x\right) < \frac{2.60}{x},$$

which is consistent with (4.4) except that we improved the upper bound for the constant C' from 5.60 to 2.60. From the following (4.5), we know that the constant C' in (4.4) actually must at least be 2.

3. For the special case of large real random $m \times m$ matrices, it has been proved in [5] that

$$\lim_{m\to\infty} P\left(\frac{\kappa_2(G_{m\times m})}{m} < x\right) = e^{-\frac{2}{x}-\frac{2}{x^2}}.$$

Therefore, we have

$$\lim_{m\to\infty} P\left(\frac{\kappa_2(G_{m\times m})}{m} > x\right) = 1 - e^{-\frac{2}{x}-\frac{2}{x^2}} \sim \frac{2}{x} \tag{4.5}$$

as $x \to \infty$. Hence, the smallest possible universal constant C in Theorem 4.5 must be no smaller than $2\sqrt{2\pi}$. Therefore, the universal constant C in Theorem 4.5 actually must satisfy

$$C \geq 2\sqrt{2\pi} \approx 5.013. \tag{4.6}$$

Similar to real random matrices, for complex random matrices, we have the following Theorem 4.6. Theorem 4.6 can be proved using the same techniques as Theorem 4.5, so we will omit the proof and only give the result.

THEOREM 4.6. *For any* $n \geq m \geq 2$ *and* $x \geq n - m + 1$*, the 2-norm condition number of* $\widetilde{G}_{m\times n}$ *satisfies*

$$P\left(\frac{\kappa_2(\widetilde{G}_{m\times n})}{n/(n-m+1)} > x\right) < \frac{1}{2\pi}\left(\frac{\widetilde{C}}{x}\right)^{2(n-m+1)},$$

where $\widetilde{C} \leq 6.298$ *is a universal positive constant independent of* $x, m,$ *and* n.

5. The lower bounds for the distribution tails. In this section, we will prove the lower bounds for the tails of the condition number distributions of random rectangular matrices whose elements are independent and identically distributed standard normal random variables. Our main results are Theorem 5.5 for real random matrices, and Theorem 5.6 for complex random matrices.

LEMMA 5.1. *For any* $B > 0$, $x > 0$*, and* $n \geq m \geq 2$*, the smallest eigenvalue* λ_{min} *of* $W_{m,n}$ *satisfies*

$$P\left(\lambda_{min} \leq \frac{B^2 n}{x^2}\right) > \sqrt{\frac{2e^{\frac{5}{6}}}{3}}\, e^{-\frac{B^2 mn}{2x^2}} \frac{1}{\Gamma(n-m+2)}\left(\frac{e^{-\frac{1}{2}}Bn}{x}\right)^{n-m+1}.$$

Proof. From the lower bound for the probability density function of λ_{min} in Lemma 3.3, we have

$$\begin{aligned}
P\left(\lambda_m \leq \frac{B^2 n}{x^2}\right) &= \int_0^{\frac{B^2 n}{x^2}} f(\lambda_m) d\lambda_m \\
&> \int_0^{\frac{B^2 n}{x^2}} L_{m,n} e^{-\frac{m}{2}\lambda_m} \lambda_m^{\frac{1}{2}(n-m-1)} d\lambda_m \\
&> L_{m,n} e^{-\frac{B^2 mn}{2x^2}} \int_0^{\frac{B^2 n}{x^2}} \lambda_m^{\frac{1}{2}(n-m-1)} d\lambda_m \\
&= L_{m,n} e^{-\frac{B^2 mn}{2x^2}} \frac{2n^{\frac{n-m+1}{2}}}{n-m+1}\left(\frac{B}{x}\right)^{n-m+1} \\
&= e^{-\frac{B^2 mn}{2x^2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)^{\frac{n-m+1}{2}}} \frac{1}{\Gamma(n-m+2)}\left(\frac{Bn}{x}\right)^{n-m+1}.
\end{aligned}$$

Note that

$$\frac{n+1}{2}\Gamma\left(\frac{n+1}{2}\right) > \sqrt{2\pi}\left(\frac{n+1}{2}\right)^{\frac{n+2}{2}} e^{-\frac{n+1}{2}},$$

and

$$\frac{m}{2}\Gamma\left(\frac{m}{2}\right) < \sqrt{2\pi}\left(\frac{m}{2}\right)^{\frac{m+1}{2}} e^{-\frac{m}{2}+\frac{1}{6m}}.$$

Therefore

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)^{\frac{n-m+1}{2}}} > e^{-\frac{n-m+1}{2}-\frac{1}{6m}}\sqrt{\frac{(n+1)^n}{m^{m-1}n^{n-m+1}}}$$
$$= e^{-\frac{n-m+1}{2}-\frac{1}{6m}}\sqrt{\frac{n^{n+1}(1+1/n)^{n+1}}{(n+1)m^{m-1}n^{n-m+1}}}$$
$$> e^{-\frac{n-m+1}{2}-\frac{1}{6m}}\sqrt{\frac{ne}{n+1}}.$$

Since $2 \leq m \leq n$, therefore, we have

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)^{\frac{n-m+1}{2}}} > \sqrt{\frac{2e^{\frac{5}{6}}}{3}}e^{-\frac{n-m+1}{2}}.$$

Therefore, we have

$$P\left(\lambda_m \leq \frac{B^2 n}{x^2}\right) > \sqrt{\frac{2e^{\frac{5}{6}}}{3}}e^{-\frac{B^2 mn}{2x^2}}\frac{1}{\Gamma(n-m+2)}\left(\frac{e^{-\frac{1}{2}}Bn}{x}\right)^{n-m+1}.$$

□

Similar to real random matrices, we have the following Lemma 5.2 for complex random matrices. Lemma 5.2 can be proved using the same techniques as Lemma 5.1, so we will omit the proof and only give the result.

Lemma 5.2. *For any $B > 0$, $x > 0$, and $2 \leq m \leq n$, the smallest eigenvalue $\widetilde{\lambda}_{min}$ of $\widetilde{W}_{m,n}$ satisfies*

$$P\left(\widetilde{\lambda}_{min} \leq \frac{B^2 n}{x^2}\right) > e^{1-\frac{1}{12m}}e^{-\frac{B^2 mn}{2x^2}}\frac{1}{\Gamma(n-m+2)^2}\left(\frac{e^{-1}B^2 n^2}{2x^2}\right)^{n-m+1}.$$

The proof of the following Lemma 5.3 is based on the upper bound of the joint probability density function of λ_{max} and λ_{min} in Lemma 3.1 and the upper bound of the incomplete Gamma function in Lemma 2.5.

Lemma 5.3. *For any $B \leq e^{-1.7}$, $x > 0$, and $2 \leq m \leq n$, the largest eigenvalue λ_{max} and the smallest eigenvalue λ_{min} of $W_{m,n}$ satisfy*

$$P\left(\lambda_{min} \leq \frac{B^2 n}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \leq x^2\right) < \frac{11B^{m-1}}{4\sqrt{4\pi}}\frac{1}{\Gamma(n-m+2)}\left(\frac{e^{-\frac{1}{2}}Bn}{x}\right)^{n-m+1}.$$

Proof. From the upper bound for the joint probability density function of λ_{max} and λ_{min} in Lemma 3.1, we have

$$P\left(\lambda_{min} \leq \frac{B^2 n}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \leq x^2\right) = \int_0^{\frac{B^2 n}{x^2}}\int_0^{tx^2} f_{\lambda_{max},\lambda_{min}}(s,t)\,ds\,dt$$
$$< C_{m,n}\int_0^{\frac{B^2 n}{x^2}}\int_0^{tx^2} e^{-\frac{1}{2}t}t^{\frac{1}{2}(n-m-1)}e^{-\frac{1}{2}s}s^{\frac{1}{2}(n+m-3)}\,ds\,dt.$$

Taking the transform $u = tx^2$, we have

$$P\left(\lambda_{min} \le \frac{B^2 n}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \le x^2\right) = C_{m,n}\left(\frac{1}{x}\right)^{n-m+1} \int_0^{B^2 n} e^{-\frac{u}{2x^2}} u^{\frac{1}{2}(n-m-1)} \left(\int_0^u e^{-\frac{1}{2}s} s^{\frac{1}{2}(n+m-3)} ds\right) du.$$

According to Lemma 2.5, if $u \le n+m-3$, then

$$\int_0^u e^{-\frac{1}{2}s} s^{\frac{1}{2}(n+m-3)} ds \le e^{-\frac{1}{2}u} u^{\frac{1}{2}(n+m-1)}.$$

Therefore, when $B \le e^{-1.7}$, we have

$$\begin{aligned} P\left(\lambda_{min} \le \frac{B^2 n}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \le x^2\right) &\le C_{m,n}\left(\frac{1}{x}\right)^{n-m+1} \int_0^{B^2 n} e^{-\frac{u}{2x^2}-\frac{1}{2}u} u^{n-1} du \\ &\le C_{m,n}\left(\frac{1}{x}\right)^{n-m+1} \int_0^{B^2 n} e^{-\frac{1}{2}u} u^{n-1} du. \end{aligned}$$

Since $B \le e^{-1.7}$, so $B^2 n \le 2(n-1)$. Applying Lemma 2.5 again, we have

$$\begin{aligned} P\left(\lambda_{min} \le \frac{B^2 n}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \le x^2\right) &\le C_{m,n}\left(\frac{1}{x}\right)^{n-m+1} e^{-\frac{B^2 n}{2}} B^{2n} n^n \\ &= \frac{e^{-\frac{B^2 n}{2}} B^{n+m-1} n^{m-1}}{4\Gamma(m-1)\Gamma(n-m+1)}\left(\frac{Bn}{x}\right)^{n-m+1}. \end{aligned}$$

From (4.2), we have

$$\frac{n^{m-2}}{\Gamma(m-1)} < \frac{e^n}{\sqrt{4\pi}}.$$

Therefore, we have

$$\begin{aligned} P\left(\lambda_{min} \le \frac{B^2 n}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \le x^2\right) &\le \frac{e^n e^{-\frac{B^2 n}{2}} B^{n+m-1} n}{4\sqrt{4\pi}\Gamma(n-m+1)}\left(\frac{Bn}{x}\right)^{n-m+1} \\ &\le \frac{B^{m-1} n(n-m+1) e^{\frac{3}{2}n} e^{-\frac{B^2 n}{2}} B^n}{4\sqrt{4\pi}} \\ &\qquad \frac{1}{\Gamma(n-m+2)}\left(\frac{e^{-\frac{1}{2}} Bn}{x}\right)^{n-m+1}. \end{aligned}$$

When $B \le e^{-1.7}$, for all $n \ge m \ge 2$, we have

$$n(n-m+1) e^{\frac{3}{2}n} e^{-\frac{B^2 n}{2}} B^n < 11.$$

Therefore,when $B \le e^{-1.7}$, we have

$$P\left(\lambda_{min} \le \frac{Bn}{x^2}, \frac{\lambda_{max}}{\lambda_{min}} \le x^2\right) < \frac{11 B^{m-1}}{4\sqrt{4\pi}} \frac{1}{\Gamma(n-m+2)}\left(\frac{e^{-\frac{1}{2}} Bn}{x}\right)^{n-m+1}.$$

□

Similar to real random matrices, we have the following Lemma 5.4 for complex random matrices. Lemma 5.4 can be proved using the same techniques as Lemma 5.3, so we will omit the proof and only give the result.

LEMMA 5.4. *For any $B^2 \le e^{-1.2}$, $x > 0$, and $2 \le m \le n$, the largest eigenvalue $\widetilde{\lambda}_{max}$ and the smallest eigenvalue $\widetilde{\lambda}_{min}$ of $\widetilde{W}_{m,n}$ satisfy*

$$P\left(\widetilde{\lambda}_{min} \le \frac{Bn}{x^2}, \frac{\widetilde{\lambda}_{max}}{\widetilde{\lambda}_{min}} \le x^2\right) < 0.0352 \frac{1}{\Gamma(n-m+2)^2}\left(\frac{e^{-1}B^2n^2}{2x^2}\right)^{n-m+1}.$$

We are now prepared to derive the lower bounds for the tails of the condition number distributions of random matrices whose elements are independent and identically distributed standard normal random variables

THEOREM 5.5. *For any $x \ge n-m+1$ and $n \ge m \ge 2$, the 2-norm condition number of $G_{m\times n}$ satisfies*

$$P\left(\frac{\kappa_2(G_{m\times n})}{n/(n-m+1)} > x\right) > \frac{1}{\sqrt{2\pi}}\left(\frac{c}{x}\right)^{n-m+1}, \tag{5.1}$$

where $c \ge 0.245$ is a universal positive constant independent of x, m, and n.

Proof. For any positive constant H, we have

$$\begin{aligned} P(\kappa_2(G_{m\times n}) > x) &= P\left(\frac{\lambda_1}{\lambda_m} > x^2\right) \\ &> P\left(\lambda_m \le \frac{H^2 n}{x^2}, \frac{\lambda_1}{\lambda_m} > x^2\right) \\ &= P\left(\lambda_m \le \frac{H^2 n}{x^2}\right) - P\left(\lambda_m \le \frac{H^2 n}{x^2}, \frac{\lambda_1}{\lambda_m} \le x^2\right). \end{aligned}$$

Let $H = e^{-1.7}$, then based on Lemma 5.1 and Lemma 5.3, we have

$$P(\kappa > x) > \left(\sqrt{\frac{2e^{\frac{5}{6}}}{3}} e^{-\frac{H^2 mn}{2x^2}} - \frac{11H^{m-1}}{4\sqrt{4\pi}}\right) \frac{1}{\Gamma(n-m+2)}\left(\frac{e^{-\frac{1}{2}}Hn}{x}\right)^{n-m+1}.$$

From Lemma 2.7, we have

$$\Gamma(n-m+2) < \sqrt{2\pi(n-m+1)}(n-m+1)^{n-m+1} e^{-(n-m+1)+\frac{1}{12(n-m+1)}}.$$

Note that, for $2 \le m \le n$, we have

$$\sqrt{n-m+1} < 1.21^{n-m+1}, \text{ and } \frac{1}{12(n-m+1)} \le \frac{1}{12},$$

Therefore, we have

$$P(\kappa_2(G_{m,n}) > x) > \left(\sqrt{\frac{2e^{\frac{5}{6}}}{3}} e^{-\frac{H^2 mn}{2x^2}} - \frac{11H^{m-1}}{4\sqrt{4\pi}}\right) \frac{e^{-\frac{1}{12}}}{\sqrt{2\pi}}\left(\frac{\frac{e}{1.21(n-m+1)} e^{-\frac{1}{2}}Hn}{x}\right)^{n-m+1}.$$

Since $H = e^{-1.7}$, $x \ge 1$, and $2 \le m \le n$, so we have

$$\left(\sqrt{\frac{2e^{\frac{5}{6}}}{3}} e^{-\frac{H^2 mn}{2x^2}} - \frac{11H^{m-1}}{4\sqrt{4\pi}}\right) e^{-\frac{1}{12}} > 0.99.$$

Therefore, we have

$$P\left(\kappa_2(G_{m,n}) > x\right) > \frac{0.99}{\sqrt{2\pi}}\left(\frac{0.248\frac{n}{n-m+1}}{x}\right)^{n-m+1} > \frac{1}{\sqrt{2\pi}}\left(\frac{0.245\frac{n}{n-m+1}}{x}\right)^{n-m+1}.$$

Therefore

$$P\left(\frac{\kappa_2(G_{m,n})}{n/(n-m+1)} > x\right) > \frac{1}{\sqrt{2\pi}}\left(\frac{0.245}{x}\right)^{n-m+1}.$$

Let $c = 0.245$, then we get (5.1). ∎

Remark:

1. The lower bound in Theorem 5.5 is for arbitrary $n \geq m \geq 2$ and $x \geq n-m+1$. For some special case of m and n, more precise lower bound can be obtained. For example, for the special case of real random $m \times m$ matrices, where $m \geq 3$, it has been proved in [2] that

$$P\left(\kappa_2(G_{m\times m}) > m \,.\, x\right) > \frac{c}{x},$$

where $c \geq 0.13$ is a universal positive constant independent of x and m.
In Theorem 5.5, however, if we take $m = n$, then we can only get

$$P\left(\kappa_2(G_{m\times m}) > m \,.\, x\right) > \frac{0.097}{x},$$

2. For the special case of real random $2 \times n$ matrices, based on the exact probability density function of $\kappa_2(G_{2\times n})$ in [5], we can get

$$P\left(\kappa_2(G_{2\times n}) > x\right) = \left(\frac{2x}{x^2+1}\right)^{n-1} \sim \left(\frac{2}{x}\right)^{n-1}$$

as $x \to \infty$. Hence, the constant c in Theorem 5.5 is no larger than 2. Therefore, the constant c in Theorem 5.5 actually satisfies

$$0.245 \leq c \leq 2. \tag{5.2}$$

Similar to real random matrices, we have the following Theorem 5.6 for complex random matrices. Theorem 5.6 can be proved using the same techniques as Theorem 5.5, so we will omit the proof and only give the result.

THEOREM 5.6. *For any $x \geq n-m+1$ and $n \geq m \geq 2$, the 2-norm condition number of $G_{m\times n}$ satisfies*

$$P\left(\frac{\kappa_2(\widetilde{G}_{m\times n})}{n/(n-m+1)} > x\right) > \frac{1}{2\pi}\left(\frac{c}{x}\right)^{2(n-m+1)},$$

where $c \geq 0.319$ is a universal positive constant independent of $x, m,$ and n.

6. The upper bounds for the expected logarithms. For square Gaussian random matrix $G_{n\times n}$, in [11], Smale asked for $E(\log\kappa_2(G_{n\times n}))$. Similarly, for rectangular Gaussian random matrix $G_{m\times n}$, it is also interesting to investigate $E(\log\kappa_2(G_{m\times n}))$. In this section, we will derive upper bounds for $E(\log\kappa_2(G_{m\times n}))$ and $E(\log\widetilde{\kappa}_2(G_{m\times n}))$. Our main results are Theorem 6.1 and Theorem 6.2.

THEOREM 6.1. *For any $n\geq m\geq 2$, the 2-norm condition number of $G_{m\times n}$ satisfies*

$$E(\log\kappa_2(G_{m\times n})) < \log\frac{n}{n-m+1} + 2.258. \tag{6.1}$$

Proof. Let $f_\kappa(x)$ be the probability density function of $\kappa_2(G_{m\times n})$, then

$$\begin{aligned} E\log\left(\frac{\kappa_2(G_{m\times n})}{6.414\frac{n}{n-m+1}}\right) &= \int_1^\infty \log\left(\frac{x}{6.414\frac{n}{n-m+1}}\right) f_\kappa(x)dx \\ &< \int_{6.414\frac{n}{n-m+1}}^\infty \log\left(\frac{x}{6.414\frac{n}{n-m+1}}\right) f_\kappa(x)dx \\ &= \int_{6.414\frac{n}{n-m+1}}^\infty P(\kappa_2(G_{m\times n})>x) \quad \frac{1}{x} \quad dx. \end{aligned}$$

From Theorem 4.3, we have

$$P\left(\kappa_2(G_{m\times n})>x\right) < \frac{1}{\sqrt{2\pi}}\left(\frac{6.414\frac{n}{n-m+1}}{x}\right)^{n-m+1}.$$

Therefore, we have

$$\begin{aligned} E\log\left(\frac{\kappa_2(G_{m\times n})}{6.414\frac{n}{n-m+1}}\right) &< \frac{1}{\sqrt{2\pi}}\int_{6.414\frac{n}{n-m+1}}^\infty \left(\frac{6.414\frac{n}{n-m+1}}{x}\right)^{n-m+1} \quad \frac{1}{x} \quad dx \\ &= \frac{1}{(n-m+1)\sqrt{2\pi}} \\ &< 0.399. \end{aligned}$$

Therefore, we have

$$\begin{aligned} E\log(\kappa_2(G_{m\times n})) &< \log\frac{n}{n-m+1} + \log 6.414 + 0.399 \\ &< \log\frac{n}{n-m+1} + 2.258. \end{aligned}$$

∎

Remark:

1. For the special case of real random $m\times m$ matrices, from the results in [12], we can get

$$E\log(\kappa_2(G_{m\times m})) \leq \log m + \frac{3+3\log 2}{2} \approx 2.54. \tag{6.2}$$

In Theorem 6.1, if we take $m=n$, then we have

$$E\log(\kappa_2(G_{m\times n})) < \log n + 2.258.$$

which is a slightly improved version of (6.2).

2. The upper bound in Theorem 6.1 is for arbitrary $n \geq m \geq 2$. For some special case of m and n or large m and n, more precise results exist:
For the special case of real random $2 \times n$ matrices, it was shown in [6] that

$$E \log(\kappa_2(G_{2\times n})) = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

For real random $m \times m$ matrices, it has been proved in [6] that

$$E \log(\kappa_2(G_{m\times m})) = \log m + c + o(1)$$

as $m \to \infty$, where $c \approx 1.537$.
For rectangular matrix $G_{m_n \times n}$, if $\lim_{n\to\infty} m_n/n = y$ and $0 < y < 1$, then it has been proved in [6] that

$$E \log(\kappa_2(G_{m_n\times n})) = \log \frac{1+\sqrt{y}}{1-\sqrt{y}} + o(1)$$

as $n \to \infty$

Similar to real random matrices, we have the following Theorem 6.2 for complex random matrices. Theorem 6.2 can be proved using the same techniques as Theorem 6.1, so we will omit the proof and only give the result.

THEOREM 6.2. *For any $n \geq m \geq 2$, the 2-norm condition number of $G_{m\times n}$ satisfies*

$$E(\log \kappa_2(\widetilde{G}_{m\times n})) < \log \frac{n}{n-m+1} + 2.240.$$

Acknowledgments. We would like to thank Julien Langou for precious discussions. We are very grateful to referees for their helpful corrections, valuable comments, and constructive suggestions which have improved the presentation of this paper.

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